ADDITIVE CONJOINT MEASUREMENT OVER INCOMPLETE ORDERS

by

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DEPARTMENT OF ELECTRICAL ENGINEERING
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CERTIFICATE

This is to certify that the work presented in this thesis titled **"ADDITIVE CONJOINT MEASUREMENT OVER INCOMPLETE ORDERS"** has been carried out by Mr. Venkatesh K Subramanian under my supervision and the same has not been submitted elsewhere for a degree.

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and style of thinking and began to appreciate the endless possibilities they indicated After this, beyond very general suggestions and hints, he let me do my own thinking as far as possible It was thus that my thesis work itself became an enriching experiance My frequent conversations with Dr. Rao and his course on the History of Scientific Ideas have broadened my horizons of thought considerably.

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Synopsis

A new model is presented for measurement over relational systems that are ordered by non-weak relations. In such systems, there can exist elements, the relation between which is undefined. While in conventional ordinal measurement theory, measurement over such incomplete orders was considered impossible, this thesis presents a scheme which, under certain conditions, and with certain assumptions, indicates a method of carrying out ordinal measurement.

The initial introduction to Measurement Theory is followed by the presentation of arguments supporting a decision criterion based upon the principle of indifference. Following this is a representation theorem that proves the feasibility of ordinal measurement over the particular class of incomplete orders that have been called *?-transitive* or *linearizable* orders. Some theorems that describe the peculiar properties of the class of linearizable orders are proved and the class of automorphisms of a ?-transitive system is identified.

Next, the theory is extended so as to apply to product (multidimensional) structures, each dimension of which is taken to be a ?-transitive system of the kind already studied. The theory is developed entirely around a set of axioms that are introduced at the start of the generalization. Several theorems are then proved that relate to the salient features of the multidimensional incomplete systems that have been named *K-structures*. A representation theorem is given for additive conjoint measurement over K-structures. It is shown that the class of K-structures is a semi-group.

Finally, an algorithm is given for the decomposition of a K-structure. This algorithm makes it possible to frame an alternative definition for a K-structure and thus completes the theory of ordinal measurement over the class of incomplete orders that are suitable for the approach outlined.

SECTION 1

Measurement Theory

1.1 HISTORICAL DEVELOPMENT

Even though Measurement Theory has been consciously developed only for the past 40 years or so, the mathematical aspects of the measurement problem were inspected somewhat earlier by Cantor(1895), Hölder(1901) and Helmholtz(1930). The earliest attempts were directed primarily towards mathematically formalizing the procedures underlying the measurement of well understood physical quantities only, to the complete exclusion of supposedly "vague" quantities of the kind encountered outside the then defined boundaries of the physical sciences. The result was that several of the quantities already well recognized at that time found themselves outside the purview of the newly born theory. Apart from the mere fact that this severely restricted the applicability of the theory, it also robbed it of the generality in form that makes a theory interesting. Campbell (1920, 1928) developed extensively the mathematical theory for the kind of measurement that is now recognized by the name 'extensive measurement' but he, too, refused to consider non-extensive

quantities as measurable in any sense at all. The primary reason for the rejection of non-extensive scales was possibly that then, unlike now, the fields of study which needed them most were still in their infancy.

FInally, the significance of the issues involved in the controversy was given due recognition, primarily due to the efforts of some eminent psychologists [Stevens], by the setting up in 1932 of a committee of the British Association for the Advancement of Science to study essentially what could, and what could not, be called measurement in the strict sense. Though the committee sat for all of eight years in search of an answer to the question, no consensus could be reached again, because the mathematicians' and physical scientists would not include non-extensive measurement as also a form of measurement. This was unacceptable to experts in other fields, especially psychology, where almost all quantities dealt with were not extensive in nature.

Over the years, however, the matter has effectively settled itself mainly due to the considerable accumulation of useful results in several fields of study, mainly psychology . What is now generally recognized as within the scope of Measurement Theory is a much wider spectrum of quantities than ever visualized earlier, and includes quantities both of the strictly physical kind, as well as the less strictly physical kind, that are encountered in a variety of spheres of human activity. In fact, the present situation is one where only the psychophysical quantities are considered to hold any serious challenge to the theorist, and hence the greater part of Measurement Theory applied here, while the more 'physical' quantities are felt to be generally well understood from a measurement theoretic viewpoint - and hence less interesting.

Historically, it is delightful to note the curious fashion in which the theory has been gradually widening its horizons, and is still actively doing so. Since the general respectability of any field of study, particularly a new one, is measured(!) by its mathematical rigour and its breadth of application, - meaning whereby, the number of occasions when people find it inevitable to appeal to its authority to bolster their arguments - it may be viewed with satisfaction that Measurement Theory is attracting the attention of experts in a multitude of diverse fields. What is presented in this Section is an overall introduction to the theory as it exists in its present form.

1.2 BASIC DEFINITIONS: REGULARITY, MEANINGFULNESS

This Subsection begins the introduction to the basic ideas that are at the root of all of Measurement Theory. Among the few texts on the subject, the ones on Measurement Theory attempt to maintain constant contact with empirical situations [Roberts, 1979] and the ones on Abstract Measurement Theory [Narens, 1985] deal entirely at an abstract level.. Publications on the subject appear usually in journals of related fields of application.

Possibly the most general definition of an act of measurement results from viewing it as the assignment of a homomorphism from the physical relational system that contains the set of values, or *states*, that the quantity to be measured may assume, together with the relevant relations and operations upon the set that we find necessary or useful to define, into the formal relational system in which we wish to carry out interpretations of the

happenings in the physical system. The choice of the formal relational system to be used is made based upon our judgment of the relative merits of different feasible systems as regards our familiarity with the use of the systems as well as their inherent flexibilities. In measurement parlance, such a homomorphism is termed a 'scale'. Sometimes, the scale is said to be 'numerical' or 'non-numerical' depending upon whether the formal system into which the homomorphism is sought is numerical or not. Following this definition henceforth, we make use of the following symbolism [Suppes & Scott, 1958] in all the notation that is to follow.

The physical relational system will be $\mathcal{A} = \langle A, R \rangle$ where R is a relation or a set of relations $\langle R_1, \ldots, R_n \rangle$ on A where R_i is an m_i-ary relation. The system $\mathcal{A}$ is said to be of type s where s is the sequence $[m_1, \ldots, m_n]$. Two relational systems are *similar* if there exists a sequence s such that both systems are of type s. The formal system $\mathfrak{B} = \langle B, R' \rangle$ will also be of type s and will be said to be *homomorphic* to $\mathcal{A}$ if there exists a function f from A into B for all $i \in \{1, \ldots, n\}$ and all sequences $\langle a_1, \ldots, a_{m_i} \rangle \in A^{m_i}$ such that

$$R_i(a_1, \ldots, a_{m_i}) \Rightarrow R'_i(f(a_1), \ldots, f(a_{m_i}))$$

$\mathcal{A}$ is said to be a *subsystem* of $\mathfrak{B}$ if $A \subseteq B$ and if for all $i \in \{1, \ldots, n\}$ R_i is a restriction of R'_i to A. $\mathcal{A}$ is said to be imbeddable in $\mathfrak{B}$ if some subsystem of $\mathfrak{B}$ is homomorphic to $\mathcal{A}$. A ***theory of measurement*** is a class $\mathfrak{K}$ of relational systems of type s closed under isomorphism for which there exists a formal relational system $\mathfrak{B}$ of the same type such that each member $\mathcal{A}$ of $\mathfrak{K}$ is imbeddable in $\mathfrak{B}$. We will only handle systems of type $s = [2]$ - which is the system that is structured by just one *binary* relation.

It is interesting to note that the above definition is quite general in nature and includes numerical scales (which are preferred almost all the time) as only one among a possibly very wide range of choices. This definition is, however, yet to find wide acceptance among people in the field. and so the innumerable possibilities that lie unexplored in the realm of non-numerical scales are as yet largely unappreciated. It may be argued that non-numerical measurement can be of no 'real' use simply *because* it is not numerical. The lack of depth in this argument can be seen as caused from a poor appreciation and understanding of why man tries to measure things in the first place. In the most general sense, measurement (meaning the assignment of homomorphisms into formal systems) is done in an attempt to understand better the physical process *using the formal system as some kind of a mental crutch to assist in comprehension*. If the human mind were better endowed than it actually is, measurement would become quite unnecessary. *Even* numerical measurement does no more than serve just this purpose. Proceeding from this quite reasonable standpoint, *any* form of measurement that achieves this end deserves to be taken seriously. However, since at the present state of development of the theory, very little literature exists on non-numerical representations, we confine all future discussions to numerical measurement.

Among the preliminary ideas used in the theory of numerical representations is the concept of a ***scale*** which is the triple $\langle \mathcal{A}, \mathfrak{B}, f \rangle$ which indicates that there exists a homomorphism $f : \mathcal{A} \to \mathfrak{B}$. The concept of the ***regularity*** of a scale is introduced as follows: if any scale $\langle \mathcal{A}, \mathfrak{B}, f \rangle$ is such that *every* other possible scale $\langle \mathcal{A}, \mathfrak{B}, g \rangle$ may be obtained from f by an transformation ϕ of f of the form $g = \phi \circ f$, then f is said to be a

regular scale. Those transformations on f (out of the set of all possible transformations) that do yield other homomorphisms from $\mathcal{A}$ to $\mathcal{B}$ are called *admissible transformations* of scale. Further, if a particular representation $\mathcal{A} \to \mathcal{B}$ has the property that *every* scale in it is regular, the representation itself is said to be regular. At the present time, most of the available results apply only to regular representations.

Any statement about a quantity being measured, if it refers to the formal system being employed must obviously make clear which particular scale is being used. Without this, the statement is usually ambiguous in its meaning and therefore useless. For example, saying that the day temperature is 60 degrees is of little use unless the temperature scale being used is also declared (or else implicitly taken to be known). However, there can also be statements that remain unambiguous in spite of being made without reference to any particular scale. When saying that today was hotter than yesterday (meaning that today's average temperature was higher than yesterday's), one is making a perfectly meaningful statement even though no particular scale is being mentioned. Thus, we find that meaningful statements are those that either,

a) make known which scale is being used or,

b) are of the kind that are invariant to scale transformations.

This concept of ***meaningfulness*** and the distinction between meaningful and meaningless statements play a central role in Measurement Theory. Examples to follow will further show that the idea of meaningfulness of scales is by no means trivial - especially in situations where our everyday preoccupation with numbers and numerical detail leads us into unwittingly arriving at totally ungrounded conclusions based upon innocent statistical data.

1.3 CLASSIFICATION OF SCALES

We now proceed to examine different *types* of regular numerical scales basing our classification not upon any physical considerations but rather solely upon considerations of their mathematical structural properties. The generality of such an approach is what makes Measurement Theory a genuine mathematical theory and therefore, quite general in its scope. We classify scales into a hierarchy of scale classes, each of which includes all those scales that are invariant to the type of admissible transformation that characterizes that class. By this scheme, we are led to distinguish 5 types of scales ordered by their increasing levels of sophistication as well as, at the same time, the increasing uniqueness of their admissible transformations. The first (and least structured) type of scale is the *nominal scale*, which merely serves to help in distinguishing the different elements of the set of states and makes no attempt to order them even in the crudest manner. This is followed by the *ordinal* scale, which already assumes that a total ordering is possible over the set of states. The third type, called an *interval* scale assumes a lot more than a mere ordering of the states - it introduces the concept of a *distance* between pairs of states, though the distance being talked about is yet strictly relative in nature, thus permitting only a comparison between pairs of states but no statements on the *absolute* 'magnitudes' of states. The fourth, the so-called *ratio* scale permits, for the first time, statements on the absolute magnitudes of the states and thus makes possible the introduction of an operation, the concatenation operation, on the states (it was indeed measurement using ratio scales that was initially recognized as the only form of *true* measurement - now known as extensive measurement). That, and

TABLE 1.3-1 CLASSIFICATION OF REGULAR SCALES

Scale type name	Admissible transformations ϕ	Examples
Nominal(I)	All one-one assignments (Distinguishing between different objects)	allotting roll numbers
Ordinal	Monotone increasing functions (Ranking objects)	utility, air pollution levels, student grades, hardness
Interval	$\{f(x) \mid f(x) = \alpha x + \beta;\ \alpha > 0\}$ (Cardinal measurement - "*how much* greater?")	*non*-absolute temperature, IQ scores(?)
Ratio	$\{f(x) \mid f(x) = (\alpha/\beta)x;\ (\alpha/\beta) > 0\}$ (Concatenation operations)	absolute temperature, mass, length, time intervals
Absolute	$f(x) = x$	counting

and most sophisticated, scale type - the *absolute* scale - is the most rigid in terms of the uniqueness of its admissible transformation - no transformations at all (from the only allowed homomorphism of mapping into the naturals) are permitted The following table C1 3-1 summarizes the classification and also provides examples of quantities that fall under the different classes

It is of interest to note that each successive type most definitely incorporates all the structural features of the preceding types We conclude this brief account on scale type with a few remarks First, we observe an apparently direct relationship between scale sophistication and the uniqueness properties of the corresponding admissible transformations - as the class of admissible transformations gets more and more restrictive, the scale becomes capable of describing the more and more complex features of the

physical systems it represents This loss of freedom in available transformations is an inevitable consequence of the higher level of structuring The other notable feature is the great gulf in terms of complexity between the nominal and the ordinal scale types While the former does without any ordering structure at all, the latter directly assumes nothing less than a totally ordered set of states This deficiency is especially tantalizing as a very large number of physical situations provide only a partially ordered system of states (or *alternatives*, as they are called in some contexts) This thesis attempts to bridge this gap by introducing a scheme which, under certain conditions, is able to carry out measurement over partially ordered, and even quasi-ordered systems of alternatives As the theory to be developed will draw considerably from the theory of ordinal scales, the following Subsection is intended to review some ordinal measurement theory in greater detail

1.4 ORDINAL MEASUREMENT

Ordinal measurement consists in the assignment of scales $\langle \mathcal{A}, \mathfrak{R}, f \rangle$ from a physical system $\mathcal{A}$ into the reals where $\mathcal{A} = \langle A, \succsim \rangle$ (or $\langle A, \succ \rangle$) and the structure $\mathfrak{R}$ is assumed to possess is $\langle R, \geq \rangle$ (or, correspondingly, $\langle R, > \rangle$) The most important characteristic feature of ordinal measurement is that the information borne by the scale does not extend beyond that contained in an ordering of objects The status of any member of the ordering is fully determined by its position alone In the latter case, $\mathcal{A}$ is said to be *strict* (the former case is *non-strict*) Conventional ordinal measurement is possible over ***total*** and ***weak*** orders - which are both ***linear*** (linear in the sense of lattice

theory) orders Total and weak orders differ in only one feature - the former are *antisymmetric* while the latter are not Antisymmetry is the property

$$x \succsim y \quad \wedge \quad y \succsim x \quad \Leftrightarrow \quad x \equiv y \qquad 1\,4\text{-}1a$$

From the fact that linear orders are characterized by negative transitivity, the above property would be rephrased for strict relations as

$$\sim(x \succ y) \quad \wedge \quad \sim(y \succ x) \quad \Leftrightarrow \quad x \equiv y \qquad 1\,4\text{-}1b$$

where $\equiv$ indicates that x and y are the same element. In a weak order, in place of identity, we would only have *equivalence* (denoted by $\approx$) Linear orders possess the property of *completeness* which is equivalent to both ways (positive *and* negative) transitivity Thus, for total orders,

$$(a \succ b) \quad \vee \quad (b \succ a), \qquad \forall \quad a \neq b \in \mathcal{A} \qquad 1\,4\text{-}2a$$

while for weak orders, a third possibility exists, a $\approx$ b thus

$$(a \succ b) \quad \vee \quad (b \succ a) \quad \vee \quad (a \approx b), \qquad \forall \quad a \neq b \in \mathcal{A} \qquad 1\,4\text{-}2b$$

For further clarification, it is convenient to summarize some other basic relationships at this point, though they may be of little immediate use

P1.4-1 Strict and non-strict relations are related as follows

a for non-strict total orders,

$$a \succsim b \quad \wedge \quad \sim(a \equiv b) \quad \Rightarrow \quad a \succ b \qquad \text{while}$$

b for non-strict weak orders,

$$a \succsim b \quad \wedge \quad \sim(a \approx b) \quad \Rightarrow \quad a \succ b$$

P1 4-2 Using 1 4-1 if necessary, we may relate total and weak orders. If $\mathcal{A}$ is a weak order, then $\mathcal{A}/\approx$ is a total order This is because

$$\sim\exists \quad a, b \in \mathcal{A}/\approx \quad \text{s t} \quad (a \approx b) \quad \wedge \quad \sim(a \equiv b)$$

In other words, a total order is just a special case of a weak order for which $|a| = 1, \ \forall \ a \in \mathcal{A}/\approx$

?

The property P1 4-2 holds not only between total and weak orders but between any pair of orders that are identical in all their features except for the former being antisymmetric and the latter not This makes it very convenient to invoke P1 4-2 whenever we wish to generalize results obtained for non-antisymmetric orders to their antisymmetric counterparts as well, provided that, of course, we are willing to forgo even a nominal scale assignment over the elements of any member of the reduction $\mathcal{A}/\approx$ Such a situation prevails, for example, between quasi orders and partial orders respectively Both these differ from the total and weak orders in the completeness property - partial and quasi orders are both *incomplete* orders, which simply means

$$\exists \quad a, b \in \mathcal{A} \quad s\,t \quad \sim(a \succ b \quad \vee \quad b \succ a \quad \vee \quad a \approx b)$$

This way of viewing a quasi order as a weak partial order will be used later in relating some results obtained for partial orders to quasi orders as well

With this background, we proceed with the study of ordinal measurement theory in greater depth As this thesis deals exclusively with finite sets and systems only, the following discussion confines itself entirely to the finite case

Essentially, we wish to determine the conditions under which a homomorphism $\mathcal{A} \rightarrow \Re$ can exist satisfying

$$a \succsim b \quad \Rightarrow \quad f(a) \geq f(b) \qquad \forall \quad a, b \in \mathcal{A}, \qquad 1\,4\text{-}3a$$

or, for $\mathcal{A}$ a strict order,

$$a \succ b \quad \Rightarrow \quad f(a) > f(b) \qquad \forall \quad a, b \in \mathcal{A}, \qquad 1\,4\text{-}3b$$

When $\mathcal{A}$ is a weak order, equivalence in $\mathcal{A}$ corresponds to equality in $\Re$,

$$a \approx b \quad \Rightarrow \quad f(a) = f(b) \qquad \forall \quad a, b \in \mathcal{A}, \qquad 1.4\text{-}3c$$

THEOREM 1.4-1 REPRESENTATION

A homomorphism satisfying 𝔇1.4-3 *can exist if and only if* $\mathcal{A}$ *is a weak order*

PROOF (Necessity) Suppose that there does exist a homomorphism as in 𝔇1.4-3 We show that $\mathcal{A}$ is a weak order We already know that a weak order is fully described by completeness - only complete orders are negatively as well as positively transitive - and possibly also asymmetric (only when the order is strict) As 𝔓1.4-1 relates strict and non-strict- and 𝔓1.4-2, total and weak-orders, we may use them to modify the proof obtained for the non-strict weak case for the other cases Thus, we first only prove that if a homomorphism of the kind 𝔇1.4-3a exists, then $\mathcal{A}$ must be weak In $\langle\Re, \geq\rangle$, we see directly that both positive and negative transitivities hold,

$$f(x) \geq f(y) \quad \wedge \quad f(y) \geq f(z) \quad \Rightarrow \quad f(x) \geq f(z)$$

which means, for the kind of homomorphism described by 𝔇1.4-3, that

$$x \succsim y \quad \wedge \quad y \succsim z \quad \Rightarrow \quad x \succsim z$$

Next, as $\sim(f(x) \geq f(y)) \quad \Rightarrow \quad \sim(f(y) \leq f(x)) \quad \Rightarrow \quad f(y) > f(x)$, and

$$f(y) > f(x) \quad \wedge \quad f(z) > f(y) \quad \Rightarrow \quad f(z) > f(x)$$

we obtain

$$\sim(x \succsim y) \quad \wedge \quad \sim(y \succsim z) \quad \Rightarrow \quad \sim(x \succsim z) \qquad \square$$

PROOF (Sufficiency) Since we are only interested in finite sets, we may adopt the following proof. let us define $f(x)$ by

$$f(x) = |x^*| \quad \text{where} \quad x^* = \{y \in A \mid x \succsim y\}$$

That this $f(x)$ satisfies 𝔇1.4-3 is directly seen For, if $x \succ y$, then $|x^*| > |y^*|$ and hence, $f(x) > f(y)$ On the other hand, if $x \approx y$, then $|x^*| = |y^*|$ and so, $f(x) = f(y)$. For a total order, $x \approx y \Rightarrow x = y$ and this yields $f(x) = f(x)$ $\square$

THEOREM 1.4-2 UNIQUENESS

81 4-3 *necessarily and sufficiently defines an ordinal scale with monotone increasing functions as the admissible transformations*

PROOF (Necessity). Since, for ordinal scales, the class of admissible transformations is the set of monotone increasing functions If ϕ is any monotone increasing function,

$$x_1 > x_2 \quad \Leftrightarrow \quad \phi(x_1) > \phi(x_2) \qquad \text{and} \tag{1 4-5a}$$

$$x_1 \geq x_2 \quad \Leftrightarrow \quad \phi(x_1) \geq \phi(x_2) \tag{1 4-5b}$$

From the above, it is clear that monotone transformations preserve the homomorphism of 81 4-3 □

PROOF (Sufficiency): We show that if any f satisfies 81 4-3, and any other $g = \phi \circ f$ *also* does, then ϕ is monotone increasing Let $a = f(x)$ and $b = f(y)$ Then, by 81 4-3,

$$a > b \quad \Leftrightarrow \quad x \succ y \quad \Leftrightarrow \quad g(x) > g(y) \quad \Leftrightarrow \quad \phi(a) > \phi(b)$$

(since g, too, satisfies 81.4-3 by assumption) From the fact that $a > b \Leftrightarrow \phi(a) > \phi(b)$, we conclude that ϕ can only be a monotone increasing function□

THEOREM 1.4-3 REGULARITY

A scale of the form 81.4-3 *is a regular representation*

PROOF: Suppose that f and g are any 2 scales satisfying 81 4-3 We show that f is regular. If this proof is of a general kind, then it holds for all f satisfying 81.4-3 and thus 81 4-3 would be a regular representation

We first prove that [Roberts & Franke, 1976] a scale f is regular if, and only if, for every other scale g, in the representation, it can be shown that

$$f(x) = f(y) \quad \Rightarrow \quad g(x) = g(y), \qquad \forall\ x, y \in \mathcal{A} \qquad 1.4\text{-}4$$

To prove necessity, if f is regular, then, by the definition of regularity, there must exist $g = \phi \circ f$ Then, $f(x) = f(y) \Rightarrow \phi \circ f(x) = \phi \circ f(y) \Rightarrow g(x) = g(y)$ To prove sufficiency, given §1.4-4, we first *define* $g = \phi[f(x)]$ Then we observe that §1.4-4 indicates that ϕ is one-one, and hence, well defined Thus, since ϕ *is* a well defined function, and since $g = \phi \circ f$, f is regular

Returning to the main proof, for any non-strict weak order $\mathcal{A}$, if, for some $x, y \in \mathcal{A}$, $f(x) = f(y)$, then, by §1.4-3, the homomorphism would require that $x \succsim y \ \wedge\ y \succsim x \Rightarrow x \approx y$, which fact, upon again applying §1.4-3, yields the result that for *any* (other) homomorphism g, *too*, $g(x) = g(y)$ This satisfies §1.4-4 and, therefore, f must be regular Finally, since the above proof is quite general, it holds for every f in the representation □

1.5 THE PHILOSOPHICAL QUESTIONS RAISED BY MEASUREMENT THEORY

At this stage, it is worth our while to ask how exactly the above mathematical treatment ought to be viewed in connection with real-life situations Most of Measurement Theory is constructed with a view of decisionmaking as the primary application of the theory. In this context, there are at least two distinct stands that are generally taken One is to view these models as *truly reflecting reality*, meaning that physical examples that do not conform to the models are themselves inherently flawed This school of thought, termed the ***normative*** approach, therefore finds it of extreme import to be able to develop *a priori* an all-encompassing theory of rationality that is able

to predict what a *rational* decision would be in every conceivable situation The fact is, however, that the most ardent efforts have failed thus far to yield results of a sufficiently general nature Unfortunately, as yet, very few negative results pointing to the impossibility of the existence of a general characterization of rational behaviour are available either Among the better known results in this connection are the nine rationality axioms of Milnor [1954], where it is shown that no existing decision criteria fit *all* the axioms Very common everyday situations provide us with examples that seem to defy even the apparently most basic and well-understood rationality criteria like transitivity [Roberts, pp 102-103] Therefore, the best that has been found possible is to define rationality under a narrow class of related situations The above objections have encouraged the development of the alternative ***descriptive*** approach in which interest is growing The basic premise here is to develop mathematical models that conform to empirical data - thus restricting theory to a role of *describing* situations in a consistent fashion without attempting to raise questions about what is rational and what is not Descriptive theory merely makes claims of the model's ability to *describe* physical situations that do agree with the model Often, however, the unavailability of models that fit exactly, and the difficulties involved in evolving whole new techniques of analysis for some situations compel one to take the normative stand [Keeney & Raiffa, 1976] As the alternative descriptive stand is more open in its approach, we propose to adopt it in our future arguments since the primary objective of Measurement Theory is, indeed, to find the mathematical models that can describe real situations sufficiently closely The present thesis is such an attempt to provide a model that, it is hoped, is less stringent in its demands on the structure of the actual physical

problem than existing models for the measurement of preference, perceived utility, and other similar subjective ordinal quantities, and therefore, presumably, more widely applicable The greater flexibility is the result of assuming an ordering of alternatives that forms an incomplete system rather than a linear order, (which was, until now, considered the minimal requirement to model such quantities) and demonstrating the possibility of ordinal measurement over it An account of how exactly this is done, what assumptions are made, and what forceful arguments supporting the assumptions can be offered, follows in the next Section

Section 3 is devoted to what may be viewed as a generalization of the results we obtain in Section 2 It is in this Section that we attend to the problem of the so-called product structures These are the multidimensional case of the ordinal measurement problem Here our results aim to generalize the existing additive conjoint measurement theory that operates only upon weak orders to a generalization that can handle a wide class of incomplete systems. The development of the theory is entirely axiomatic and ends with a representation theorem that proves the possibility of additive conjoint measurement over incomplete orders

The penultimate Section, Section 4 goes into the specific problem of inventing an algorithm that enables the analysis of the multidimensional incomplete relational structures that were introduced in Section 3 It is explained that the algorithm becomes necessary because the class of incomplete structures dealt in Section 3 forms a semi-group (and therefore, the members of the class do not have inverses) It thus completes the theoretical model, the development of which is the objective of this thesis

Section 5 concludes the thesis by discussing in hindsight the results obtained The general approach taken in developing the model is viewed critically and the assumptions made are subjected to study The possibilities of proceeding on a different set of assumptions than those adopted by us are looked into and the peculiarities of the results we have obtained are commented upon

In this thesis, the following system of notation has been used The total thesis is divided into Sections, each of which contains several Subsections that are numbered sequentially within each Section Expressions, figures, properties, tables, theorems, and algorithms are numbered sequentially within each Subsection and are identified by appropriate prefixes in script letters ($\mathcal{A}$ for algorithms, $\mathcal{C}$ for tables, $\mathcal{E}$ for expressions, $\mathcal{F}$ for figures, $\mathcal{P}$ for properties and $\mathcal{T}$ for theorems) Thus, $\mathcal{A}3$ 4-1 is the first algorithm in Section 3, Subsection 4, $\mathcal{T}2$ 3-2 is the second theorem in Section 2, Subsection 3, etc Further, closely related expressions are assigned the same number and are distinguished only by different lower case suffixes as, for example, in $\mathcal{E}4$ 3-5*a* and $\mathcal{E}4$ 3-5*b* All but the most well known symbols are explained at the points where they are used for the first time

SECTION 2

Linearizable Structures

2.1 *INTRODUCTION*

This Section deals with the specific issue of ordinal measurement over incompletely ordered sets To begin with, the arguments supporting the assumptions made in introducing such a scheme are presented along with the essential features of the scheme itself It is pointed out that the assumption of an attitude of indifference is possibly the best that can be done when one is faced with a situation where a choice is to be made among a pair of alternatives, information about which is unavailable Clearly, however, this is not a contention that can be supported by anything more than powerful arguments, one cannot provide anything in the nature of a logical proof. But once the stand taken is accepted, a whole body of interesting and consistent results are shown to logically follow This Section, therefore, contains a statement of the particular conditions under which ordinal measurement can be attempted over incomplete orders and a proof of the necessity and sufficiency of the conditions This is followed by a discussion of the distinctive

properties of such so-called *linearizable* relational systems. After this is presented an algorithm for testing whether or not a given structure is linearizable A slight modification of the same algorithm yields an algorithm for actual linearization of a structure that is known to be linearizable The actual representation and uniqueness theorems for the measurement problem are so presented as to show that the case of the incomplete linearizable systems can be reduced to a form in which the same theorems that are applied to total orders, namely, $\mathfrak{T}$1 4-1, $\mathfrak{T}$1 4-2 and $\mathfrak{T}$1 4-3, can be directly used

2.2 THE GENERAL INCOMPLETE SYSTEM

We shall present all our arguments in this Section with the problem of measurement of an individual's preference in view The classical preference problem is formulated as follows an individual is presented with a set A, of alternatives, and is assigned the task of expressing his preferences He is to express his choice by considering one pair of alternatives at a time, and specifying for each such pair of elements in A (or, equivalently, for each element in A^2), which one of the 3 possibilities applies

- he prefers the first to the second
- he prefers the second to the first
- he is indifferent to both

When the choice specification is completed for each pair, classical preference measurement theory provides methods for the assignment of an ordinal scale provided internal inconsistencies such as violations of transitivity are absent

Generally speaking, the above scheme is quite adequate whenever the possibility of the individual being unable to choose from any of the 3 decisions given above does not exist This may indeed be the case in a very large class of physical situations However, there do often arise instances when, for some particular pairs of alternatives, the individual is genuinely unable to decide for some reason, either because any attempt to force a comparison may be absurd in physical terms ("What would you like, a book or a picnic?" is an example) or due to a lack of knowledge of the consequences that the alternatives he is to choose from entail This is the case of the general incompletely ordered relational system Classical preference measurement theory does not recognize that such a state of affairs can prevail and therefore, does not attempt to solve this preference measurement problem The present thesis evolves a method by which ordinal measurement is still possible in a particular subclass, called *linearizable* systems, of the class of incomplete systems - provided a particular decision criterion is allowed to be used

As stated above, a possible reason for an individual's inability to make a choice from among a pair of alternatives may be that he does not have information regarding the consequences that may ensue from his choice This situation has a parallel in statistical decision theory in the problem of decisionmaking under uncertainty There, a decision needs to be made in the absence of information of even the probabilities with which events that determine the final consequences are likely to happen The parallel that exists in the theory of finite games [Suppes, 1961] is of a game that may be represented by a matrix in which the player must select a row and the opponent selects a column The entries in the matrix represent the gains that accrue to

the player depending upon which row he chooses and which column the opponent selects The examples quoted above involve situations where the individual has to decide on a course of action when he is not in possession even of statistical information of the opponent's possible move

The principle of indifference due to Laplace states that in a situation where we are faced with the need to make a decision in the absence of information about which move the opposing agent is likely to make out of a set of possible moves - even information about the probabilities with which moves are chosen by the opponent - we must assume equal probabilities on the set of moves of the opponent and then choose *our* move so as to minimize the expected cost of the decision In the case of the game described earlier, the player applying the indifference principle would assign equal probabilities to all the columns and then choose the row that minimizes the expected cost

The decision criterion proposed above is also sometimes called the principle of *insufficient reason* [Cox, 1961] This is a criterion of judgment of equal probabilities when either the data provides no information or when the information provided is equivocal with respect to all the alternatives The principle of insufficient reason stands in contrast to the principle of *sufficient* reason that is invoked when all alternatives are *known* to have equally favourable consequences In either situation (of complete or of no information) one ends up viewing all the alternatives with an impartial eye, in the fully determined case, because there is no ground for doubt that any particular alternative is less attractive than any other, and in the latter case, because there is no ground for preferring any particular alternative to any other as all are equally mysterious

The relevance of the above discussions to our problem of ordinal measurement over incompletely ordered relations is that there exists a very obvious resemblance to the kind of problems faced above In one, we assign equal probabilities to alternatives due to the absence of information regarding the consequences implied by them, and in the other, we assign *equal ordinal numbers* to different alternatives among which we are unable to make a choice for either the reason that we have insufficient information or because the alternatives are themselves not *comparable* This approach, as stated in the introduction, is not something that can be supported by *logic* All we can say is that this is one possible attitude that we may take which has a reasonably sound basis for use in this and other related situations

At this stage, it is important to reiterate that not all incomplete orders are amenable to the application of this principle The class of incomplete orders to which we may apply the principle without arriving at inconsistent or contradictory results is to be called the class of ***linearizable*** incomplete structures The requirements on an incomplete structure that ensure its linearizability are stated and proved in the next Subsection We restrict all our future discussions solely to linearizable structures and their derivatives

2.3 THE ?-TRANSITIVE INCOMPLETE SYSTEM

We now begin to consider the particular type of incomplete system that lends itself to our form of measurement We are interested in a type [2] relational system $\mathcal{A} = \langle A, \succ \rangle$ where the binary ordering relation $\succ$ is non-weak (is negatively antisymmetric) and has the following features

- irreflexivity,

$$\sim(a \succ a) \qquad \forall\ a \in \mathcal{A}$$

- asymmetry,

$$a_i \succ a_j \Rightarrow \sim(a_j \succ a_i) \qquad \forall\ a_i, a_j \in \mathcal{A}$$

- transitivity,

$$(a_i \succ a_j) \wedge (a_j \succ a_k) \Rightarrow a_i \succ a_k \qquad \forall\ a_i, a_j, a_k \in \mathcal{A}$$

- incompleteness

$$\exists\ a_i, a_j \in \mathcal{A} \quad s\,t \quad \sim(a_i \succ a_j) \wedge \sim(a_j \succ a_i) \wedge \sim(a_i \equiv a_j)$$

We let the incompleteness property induce the $?$ relation on $\mathcal{A}$ by the following definition

$$\sim(a_i \succ a_j) \wedge \sim(a_j \succ a_i) \wedge \sim(a_i \equiv a_j) \quad \Leftrightarrow \quad a_i \mathbin{?} a_j$$

The new relation $?$ is seen to possess

- irreflexivity,

$$\sim(a \mathbin{?} a) \qquad \forall\ a \in \mathcal{A}$$

- symmetry,

$$a_i \mathbin{?} a_j \Rightarrow a_j \mathbin{?} a_i \qquad \forall\ a_i, a_j \in \mathcal{A}$$

The system $\langle A, \succ, ? \rangle$ is now complete in the sense that

$$(a_i \succ a_j) \vee (a_j \succ a_i) \vee (a_i \mathbin{?} a_j) \qquad \forall\ a_i \neq a_j \in \mathcal{A} \qquad 2\,3\text{-}1$$

To avoid any possible confusion, we would do well to clearly state how any element of $\mathcal{A}$ is related to *itself* when the ordering relation is strict Such a situation of a_i, a_j being identical is specified by

$$a_i \equiv a_j \quad \Leftrightarrow \quad \sim(a_i \succ a_j) \wedge \sim(a_j \succ a_i) \wedge \sim(a_i \mathbin{?} a_j)$$

When the ordering relation is reflexive (non-strict) and non-weak, asymmetry and negative antisymmetry would be replaced by antisymmetry and negative asymmetry Now, the $?$ relation would be induced by

$$\sim(a_i \succeq a_j) \;\wedge\; \sim(a_j \succeq a_i) \;\Leftrightarrow\; a_i \,?\, a_j$$

The ? relation would still be irreflexive and symmetric The case of identity when the ordering relation $\succeq$ is non-strict would be specified by

$$a_i \equiv a_j \quad \Leftrightarrow \quad a_i \succeq a_j \quad \wedge \quad a_j \succeq a_i$$

Clearly, the ? relation is neither an ordering relation (as it is neither positively nor negatively asymmetric) nor is it an equivalence relation (as it is not reflexive) However, such relational structures *are* often encountered in everyday situations, one ready example being the "brother of" relation We next present the representation theorem for ?-transitive systems

THEOREM 2.3-1 REPRESENTATION

A homomorphism from an irreflexive, antisymmetric, transitive, incomplete relational system $\mathcal{A} = \langle A, \succ, ? \rangle$ ***into the reals,*** $\Re = \langle R, > \rangle$ ***satisfying***

$$a_i \succ a_j \quad \Rightarrow \quad f(a_i) > f(a_j) \qquad 2.3\text{-}2a$$

exists, provided we impose the assignment

$$a_i \,?\, a_j \quad \Leftrightarrow \quad f(a_i) = f(a_j) \qquad 2.3\text{-}2b$$

if, and only if the ? relation is transitive,

$$(a_i \,?\, a_j) \;\wedge\; (a_j \,?\, a_k) \;\Rightarrow\; a_i \,?\, a_k \qquad \forall \; a_i, a_j, a_k \in \mathcal{A} \qquad 2.3\text{-}3$$

PROOF: We start the proof by defining a ***linearizing reduction*** of $\mathcal{A}$ which will reduce $\mathcal{A}$ to the new system ${}^1\mathcal{A} = \langle {}^1A, \dot{\succ} \rangle$ with the following features

- ${}^1A$ is a partition of A
- the relation $\dot{\succ}$ is defined as follows

$${}^1a_i \;\dot{\succ}\; {}^1a_j \;\Leftrightarrow\; a_i \succ a_j \qquad \forall \; a_i \in {}^1a_i,\; a_j \in {}^1a_j,$$
$$\forall \; {}^1a_i \;\dot{\succ}\; {}^1a_j \in {}^1A \qquad 2.3\text{-}4$$

The new relation $\dot{\succ}$ is seen to be strict (irreflexive), asymmetric, negatively antisymmetric, positively as well as negatively transitive and complete - which imply a total order

It is important to note that a linearizing reduction of the form described above *always* exists At worst, ${}^1A$ will simply be the identity partition ${}^1A = \{(A)\}$ - which would mean that $|{}^1A| = 1$, and at best, when $\succ$ is complete in $\mathcal{A}$ (which happens when ? is empty in $\mathcal{A}$), the finest partition, with $|{}^1A| = |A|$ At this stage, it is enough to appreciate the fact that such a partitioning is always possible, the matter of the cardinality of the resulting partition and the algorithm that makes such a partitioning possible are not of immediate relevance

Now, we recognize that for each ${}^1a \in {}^1A$, there is an internal structure which is just the restriction of $\succ$ and ? to ${}^1a \subset A$, of these, either, or both, or neither may be empty in $\langle {}^1a, \succ, ? \rangle$ We denote ${}^1\mathcal{A}$, the *linearized reduction* of $\mathcal{A}$, as

$$ {}^1\mathcal{A} = \mathcal{L}red(\mathcal{A}) $$

From our earlier discussion, we know that

$$ 1 \leq |\mathcal{L}red(\mathcal{A})| \leq |\mathcal{A}|. $$

We next effect a linearizing reduction on each $\langle {}^1a, \succ, ? \rangle$ to obtain *their* partitions

$$ \mathcal{L}red(\langle {}^1a_i, \succ, ? \rangle) = \bigcup_{k=1}^{|x_i|} \langle {}^2a_{ik}, \succ, ? \rangle $$

Again,

$$ 1 \leq |\mathcal{L}red(\langle {}^1a_i, \succ, ? \rangle)| \leq |{}^1a_i| \qquad \forall \quad \langle {}^1a_i, \succ, ? \rangle \in {}^1\mathcal{A} $$

Let

$$^{2}\mathcal{A} \quad \triangleq \quad \mathcal{L}red(\mathcal{L}red(\mathcal{A})) \quad = \quad \bigcup_{i=1}^{|x_i|} \mathcal{L}red(\langle {}^{1}a_i, \succ, ?\rangle)$$

From the above, and from the inequalities

$$1 \;\leq\; |\mathcal{L}red(\mathcal{A})| \;\leq\; |\mathcal{A}|$$

$$1 \;\leq\; |\mathcal{L}red(\langle {}^{1}a_i, \succ, ?\rangle)| \;\leq\; |{}^{1}a_i| \qquad \forall \quad \langle {}^{1}a_i, \succ, ?\rangle \;\in\; {}^{1}\mathcal{A}$$

we conclude that

$$|{}^{1}\mathcal{A}| \;\leq\; |{}^{2}\mathcal{A}| \;\leq\; |\mathcal{A}| \qquad 2\,3\text{-}5$$

This second linearizing reduction has yielded $^{2}\mathcal{A} = \langle {}^{2}A, \dot{\succ}_i \rangle$, which again is a strict total order Since

$$\mathcal{L}red(\langle {}^{1}a_i, \succ, ?\rangle) \;=\; \bigcup_{k=1}^{|x_i|} \langle {}^{2}a_{ik}, \succ, ?\rangle$$

we also have the inequality

$$\underset{k}{Max}|{}^{2}a_{ik}| \;\leq\; |{}^{1}a_i| \qquad 2\,3\text{-}6$$

We carry out further linearizing reductions in the same manner as outlined above to obtain $\mathcal{A} \rightarrow {}^{1}\mathcal{A} \rightarrow {}^{2}\mathcal{A} \rightarrow \quad \rightarrow {}^{f}\mathcal{A}$ The final reduction, ${}^{f}\mathcal{A}$ is characterized by the property

$$|\mathcal{L}red(\langle {}^{f}a_i, \succ, ?\rangle)| \;=\; 1 \qquad \forall \quad \langle {}^{f}a_i, \succ, ?\rangle \;\in\; {}^{f}\mathcal{A}$$

A consequence of the above property is that

$$|{}^{f+1}\mathcal{A}| \;=\; |{}^{f}\mathcal{A}|$$

Extrapolating §2 3-5, we obtain

$$|{}^{1}\mathcal{A}| \;\leq\; |{}^{2}\mathcal{A}| \;\leq\; . \;\; \leq |{}^{f}\mathcal{A}| \;\leq\; |\mathcal{A}|$$

This indicates a situation in which we are unable to effect any further non-trivial linearizing reductions upon *any* of the elements of ${}^{f}\mathcal{A}$, we say of ${}^{f}\mathcal{A}$ that it is a *maximally linearized reduction* of $\mathcal{A}$

Like all its predecessors ${}^{g}\mathcal{A}$, $g \in \{1, 2, \ldots, f\}$, ${}^{f}\mathcal{A}$ is also a strict total order. Therefore, ℱ1 4-1 to ℱ1 4-3 are directly applicable on $\mathcal{A}$. Moreover, the above construction can be modified to handle a non-strict ordering relation using 𝒫1 4-1. Weak (non-antisymmetric) ordering relations can also be treated with the help of 𝒫1 4-2, provided we are willing to tolerate the assignment

$$ {}^{f}a_i \approx {}^{f}a_j \quad \Leftrightarrow \quad f({}^{f}a_i) = f({}^{f}a_j) $$

which prevents the assignment of a nominal scale.

With this, the problem of ordinal measurement over ${}^{f}\mathcal{A}$ is solved as it satisfies all the classical ordinal measurement axioms. It is now only left to show the connection from this to the ordinal measurement problem over incomplete systems when a solution is desired of the form 82 3-2 and to prove the necessity and sufficiency of 82 3-3 for such a scale to exist.

Consider any $\langle {}^{f}a_i, \succ, ?\rangle \in {}^{f}\mathcal{A}$ for which $|{}^{f}a_i| > 1$. In this structure $\langle {}^{f}a_i, \succ, ?\rangle$, we are assured, by 82 3-1, that at most one (but never both) of $\succ$ and $?$ is empty in the restriction. Now, since each $\langle {}^{f}a_i, \succ, ?\rangle$ is not further linearizable by assumption, we are assured that $?$ is not empty in any $\langle {}^{f}a_i, \succ, ?\rangle$ (because, if it were, then $\langle {}^{f}a_i, \succ, ?\rangle$ would be a total order with $|{}^{f}a_i| > 1$, and therefore, further reducible in the manner of 82 3-4, thus contradicting the assumption that $|\mathcal{L}red(\langle {}^{f}a_i, \succ, ?\rangle)| = 1$). Now that we have proved the non-emptiness of $?$, we are left with only two possibilities - either the emptiness, or non-emptiness of $\succ$ in $\langle {}^{f}a_i, \succ, ?\rangle$. It is now shown that a scale of the kind dictated by 82 3-2 cannot exist if $\succ$ is non-empty in $\langle {}^{f}a_i, \succ, ?\rangle$. Suppose that there is a $\langle {}^{f}a_i, \succ, ?\rangle$ with $|{}^{f}a_i| > 2$ ($|{}^{f}a_i| > 2$ is the minimum needed to ensure that $\succ$ is non-empty when we are given that $?$ is

already so) As $\succ$ is assumed non-empty in $\langle {}^f a_i, \succ, ? \rangle$

$$\exists \; x, y \in {}^f a_i \quad \text{s t} \quad x \succ y$$

As $?$ is also assumed non-empty in $\langle {}^f a_i, \succ, ? \rangle$,

$$\exists \; z \in {}^f a_i \quad \text{s t} \quad (x \; ? \; z) \vee (y \; ? \; z)$$

This allows for several possible cases, none of which is capable of satisfying 82 3-2 as is shown below

- $(x \succ y) \wedge (y \succ z) \wedge (x \; ? \; z)$
- $(x \succ y) \wedge (z \succ x) \wedge (y \; ? \; z)$

These violate the transitivity of $\succ$ and therefore cannot exist

- $(x \succ y) \wedge (x \succ z) \wedge (y \; ? \; z)$
- $(x \succ y) \wedge (z \succ y) \wedge (x \; ? \; z)$

These can be further linearized to yield $\{(x), (y, z)\}$ and$\{(x, z), (y)\}$ respectively and therefore contradict the assumption that $\langle {}^f a_i, \succ, ? \rangle$ is already maximally linearized Therefore, they cannot exist in ${}^f\mathcal{A}$

- $(x \succ y) \wedge (y \; ? \; z) \wedge (x \; ? \; z)$

This is the only truly interesting case as its existence is rendered impossible, not by assumptions made or conditions imposed incidentally during our discussions but by the central feature of our scheme, namely 82 3-2 To prove the claim of 82 3-3, we assume the contrary and arrive at a contradiction which as we now show, may be avoided only by accepting 82 3-3 Suppose that a scale assignment of the form of 82 3-2 does exist, then

$$x \succ y \quad \Rightarrow \quad f(x) > f(y)$$

$$y \; ? \; z \quad \Rightarrow \quad f(y) = f(z) \quad \text{and}$$

$$x \; ? \; z \quad \Rightarrow \quad f(x) = f(z)$$

which is a contradiction This contradiction would clearly not occur

only if $\succ$ were empty in $\langle {}^f a_i, \succ, ?\rangle$ Then,

$$(x \; ? \; y) \;\wedge\; (y \; ? \; z) \;\wedge\; (x \; ? \; z)$$

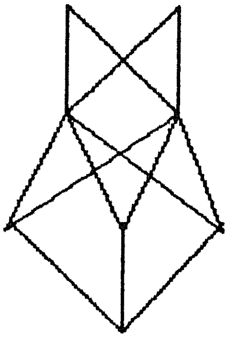
which is just a transitivity condition on $?$ in $\langle {}^f a_i, \succ, ?\rangle$ Finally, under these conditions, we may obtain an ordinal scale on $\mathcal{A}$ itself from the ordinal scale already available on the total order ${}^f\mathcal{A}$ by stipulating that

$$f(a) = f({}^f a_i) \qquad\qquad \forall \; a \in {}^f a_i$$

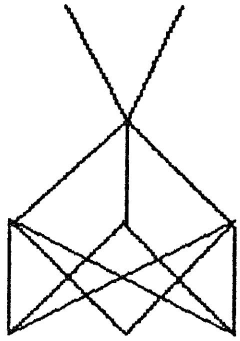
Since $?$ is empty in $\langle {}^f a_i, \succ, ?\rangle$, the above amounts to §2 3-3 This proves the necessity of ?-transitivity Sufficiency follows from the fact that the proof is constructive □

From now on, we shall call the class of quasi orders that lend themselves to the type of scale described by §2 3-2 ***linearizable*** or, in view of ¶2.3-1, ***?-transitive*** structures ¶2 3-1 shows the Hasse diagrams of some linearizable structures and ¶2 3-2 attempts to diagrammatically express the process of progressive linearization of a ?-transitive structure An idea that shall be invoked in future is that of a ***level*** of a linearizable structure A level ℓ_i of a linearizable structure is just the name given to each $\langle {}^f a_i, \succ, ?\rangle$ $\in {}^f\mathcal{A}$, the maximally linearized reduction of $\mathcal{A}$ As ${}^f\mathcal{A} = \langle {}^f A, \dot{\succ}\rangle$ is a total order, we have a totally ordered sequence of levels which we shall name sequentially so as to satisfy $\ell_i \;\dot{\succ}\; \ell_j \Leftrightarrow i > j$

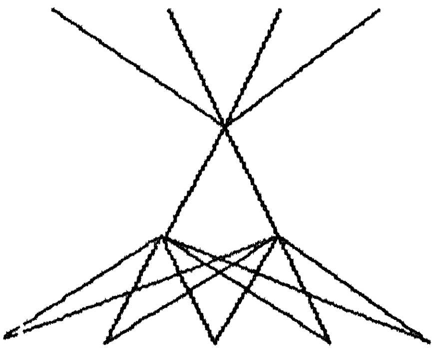
Finally, as ¶2 3-1 has proved conclusively that $\succ$ is empty in each $\langle {}^f a_i, \succ, ?\rangle$, we may as well abbreviate its notation to $\langle {}^f a_i, ?\rangle = \langle \ell_i, ?\rangle$ We term the quantity $|{}^f a_i| \triangleq |\ell_i|$, the *width* of a level The above facts show that all the information about a linearizable structure is captured the



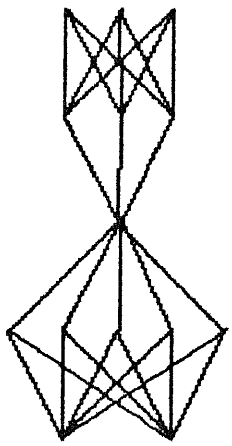
[2 2 3 1]



[2 1 3 3]



[4 1 2 5]



[3 3 1 5 2]

Fig.2.3–1: Examples of Linearizable Structures

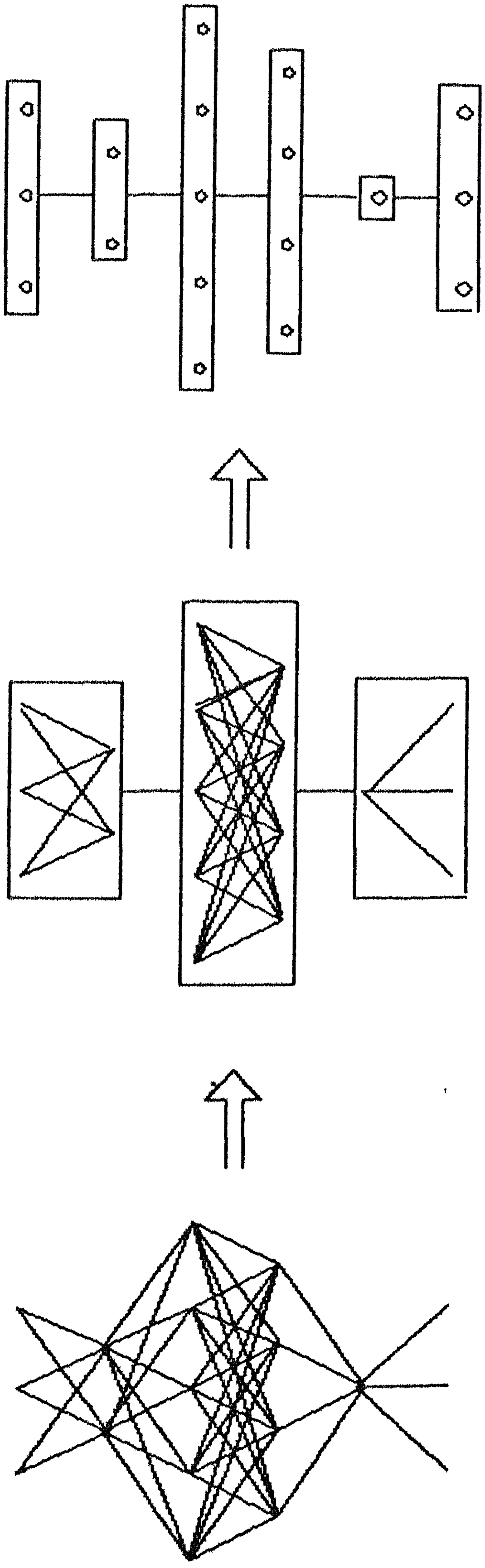


Fig.2.3–2: Progressive Linearization of a ?–transitive structure

widths of the levels and the sequence in which they occur and permit us to represent completely a linearizable structure by just an array of numbers as

$$\mathcal{A} \equiv [|\ell_1|, |\ell_2|, \quad , |\ell_\lambda|]$$

where $|^f A| = \lambda$ We now define a *maximal chain* of a linearizable lattice structure as a system of elements

$$a_1 \succ a_2 \succ \quad \succ a_\lambda \qquad s\ t \qquad a_i \in \ell_i$$

THEOREM 2.3-2 LEVEL DISTINCTNESS

All maximal chains in a linearizable structure have the same length equal to the number of levels

PROOF This follows from the fact that a linearizing reduction of a structure is a partition of it As $\mathcal{A} \to {}^1\mathcal{A} \to {}^2\mathcal{A} \to \quad \to {}^f\mathcal{A}$ is a sequence of linearizing reductions, ${}^f\mathcal{A} = \{\ell_1, \ell_2, \quad , \ell_\lambda\}$ is also a partition of $\mathcal{A}$ Therefore

$$\sim\exists\ a \in \mathcal{A} \quad s\ t \quad {}_i a \succ a \succ {}_{i+1}a \quad {}_i a \in \ell_i, \quad {}_{i+1}a \in \ell_{i+1}$$

Therefore, the longest chains (maximal chains) of elements would be of the form

$${}_1a \succ {}_2a \succ \cdot \quad \succ {}_\lambda a \qquad\qquad {}_i a \in \ell_i, \quad i \in \{1, \quad , \lambda\} \quad \Box$$

THEOREM 2.3-3 AUTOMORPHISMS

The set of one-one maps of each level onto itself constitutes the class of automorphisms of a linearizable structure.

PROOF: Suppose that we have a structure obtained from $\mathcal{A}$ by mapping each level onto itself We name $a_j \succ a_i$ as the relation $a_i \prec a_j$ and the relation $\ell_m \dot{\succ} \ell_n$ as $\ell_n \dot{\prec} \ell_m$ If $a_i \in \ell_n$ and $a_j \in \ell_m$, and, correspondingly, $a^*_i \in \ell^*_n$ and $a^*_j \in \ell^*_m$, then we can show that the structure, $\mathcal{A}^*$ is isomorphic to $\mathcal{A}$

$$a_i \succ a_j \quad \Leftrightarrow \quad \ell_n \dot{\succ} \ell_s \quad \Leftrightarrow \quad \ell^*_n \dot{\succ} \ell^*_s \quad \Leftrightarrow \quad a^*_i \succ a^*_j$$

Similarly, if $a_i \prec a_j$

$$a_i \prec a_j \quad \Leftrightarrow \quad \ell_n \dot{\prec} \ell_s \quad \Leftrightarrow \quad \ell^*_n \dot{\prec} \ell^*_s \quad \Leftrightarrow \quad a^*_i \prec a^*_j$$

Finally,

$$a_i \ ?\ a_j \quad \Leftrightarrow \quad \ell_n \equiv \ell_s \quad \Leftrightarrow \quad \ell^*_n \equiv \ell^*_s \quad \Leftrightarrow \quad a^*_i \ ?\ a^*_j$$

Therefore,

$$a_i \circ a_j \quad \Leftrightarrow \quad a^*_i \circ a^*_j \qquad \forall\ a_i, a_j \in \mathcal{A}, \quad \forall\ a^*_i, a^*_j \in \mathcal{A}^*, \quad \forall\ \circ \in \{\succ, \prec, ?\}$$

□

Cardinality of the class of automorphisms: The size of the class of automorphisms of a given ?-transitive structure $\mathcal{A} = [\ell_1, \ell_2, \ \ , \ell_\lambda]$ may be computed as follows all permutations of the elements of a level yield automorphisms of $\mathcal{A}$, therefore, we have $(|\ell_i|)!$ possibilities for each ℓ_i Thus the ?-transitive structure will have a total of $\prod_{i=1}^{\lambda}(|\ell_i|)!$ automorphisms

2.4 ALGORITHMS

This Subsection is devoted to the presentation of two closely related algorithms The first tests whether a given structure $\mathcal{A}$ is linearizable (or ?-transitive, which is the same thing, as shown by ¶2.3-1). The second algorithm, when used along with a sorting algorithm, enables us to directly linearize a ?-transitive structure so that an ordinal scale of the kind given in §2.3-2 may be assigned In fact, both algorithms are essentially different applications of the same principle, modified to achieve different ends

Before we begin describing the algorithms themselves, we clarify the conventions employed in the array representation of a relational structure that has been used here Any structure with n elements is fully described by a two dimensional array of $n \times n$ elements Each entry in the array is addressed by an ordered pair of positive integers - the first of which is the ordinal number of the row containing the entry and the latter, the column The entry at location (i, j), denoted by $[i, j]$, details the relationship the element a_i bears to the element a_j The entries are therefore one of only four possible symbols

- $a_i \equiv a_j \Rightarrow [i, j] = '\equiv'$ This occurs all along the diagonal and nowhere else
- $a_i \succ a_j \Rightarrow [i, j] = '\succ'$ Then, invariably, $[j, i] = '\prec'$
- $a_i \prec a_j \Rightarrow [i, j] = '\prec'$ Then, invariably, $[j, i] = '\succ'$
- $a_i \, ? \, a_j \Rightarrow [i, j] = ?$ By symmetry of ?, $[j, i] = '?'$

Thus, if $a_5 \succ a_7$, we have $[5,5] = [7,7] = '\equiv'$, $[5,7] = \succ$, and $[7,5] = \prec$ A typical structure array for a structure with $n = 5$ would be as shown in C2 4-1

We now take a look at the features the structure array of a ?-transitive structure must possess

- As ? is symmetric, the locations of ? must be symmetric about the principal diagonal Thus, $[i, j] = ? \Leftrightarrow [j, i] = ?$
- From ?-transitivity, we get the requirement that if the j_1th row contains ? at columns $j_2, j_3, \ldots j_n$, then the j_2th, j_3th, . . , j_nth rows must also necessarily contain ? at the j_1th, j_3th, , j_nth, columns, the j_1th, j_2th, .. , j_nth, columns and the j_1th, j_2th, , j_{n-1}th, columns respectively To put it another way, the relation $(? \cup \equiv)$ is an equivalence relation

TABLE 2.4-1 STRUCTURE ARRAY EXAMPLE

	1	2	3	4	5
1	$\equiv$	$\succ$	$\succ$	$\succ$	$\succ$
2	$\prec$	$\equiv$	?	$\succ$	?
3	$\prec$	?	$\equiv$	?	$\succ$
4	$\prec$	$\prec$	?	$\equiv$	?
5	$\prec$	?	$\prec$	?	$\equiv$

ALGORITHM 2.4-1 ?-TRANSITIVITY TEST

Begin { ?-transitivity test }

Repeat

1. Initialize i to 1 where i denotes the row number

2 Construct $J(i) = \{j\}$, the set of columns j s t

$[i,j] = (? \vee \equiv)$

3 if $J(i) \neq \emptyset$ then

construct $J(k)$, $\forall\ k \in J(i)$

Test whether

a $J(i) \neq J(k)$, $\forall\ k \in J(i)$

b. $J(i) \cap J(k) \neq \emptyset$, $\forall\ k \notin J(i)$.

if (a or b) then

?-transitivity = false,

exit,

4. $i = i + 1$

Until $i = n$

?-transitivity = true

Stop { ?-transitivity test }

ALGORITHM 2.4-2 LINEARIZATION

Begin { linearization }

Repeat

1 Initialize i to 1 where i denotes the row number and initialize D to $\emptyset$

2 Construct $J(i) = \{j\}$, the set of columns j s t $[i,j] = (? \vee \equiv)$

3 $i = i + 1$

4 if $i \in J(k)$ where $(k \notin D) \wedge (1 \leq k \leq i - 1)$ then

$D = D \cup \{i\}$

go to 3

else go to 2

Until: $i = n$

5 Construct ${}^{f}A = \{J(i)\}$,

Stop { linearization }

The result of $\mathcal{A}$2 4-2 is the set ${}^{f}A = \{J(i)\}$, which is the set of levels (ℓ_i) of the given linearizable structure $\mathcal{A}$ The algorithm does not effect an ordering of the levels But as $\langle \ell_i, \dot{\succ} \rangle$ is a total order, any of the several well known sorting algorithm can be used to order the levels

2.5 SUMMARY

With this we complete the discussion of the problem of ordinal measurement over incompletely ordered systems that possess the property of ?-

transitivity The theorems stated and proved in this Section introduce, for the first time, this special class of incomplete systems and characterize their salient properties like level distinctness The theorem about the class of automorphisms of a ?-transitive structure demonstrates that the array representation of a ?-transitive system (introduced even before the theorem) is sufficient to characterize a ?-transitive structure uniquely to within an automorphism (this is so as all automorphisms of a given ?-transitive structure have exactly the same array representation) Finally, the algorithms $\mathcal{A}$2 4-1 and $\mathcal{A}$2 4-2 provide means for testing for ?-transitivity and linearizing ?-transitive systems

The problem dealt with in this Section may be termed the one-dimensional ?-transitive structure linearization problem This is one-dimensional in the sense that we assume no more structure on the set of alternatives than that dictated by $\succ$ and ? It is often possible in real-life situations to find (as the descriptive approach attempts to do), or impose (the normative approach) other, more subtle, structural properties The assumption of the existence of those structural properties for incomplete structures that permit conjoint measurement allows us to generalize our present results considerably. They then also apply to so-called product - or multidimensional - relational structures, and thus present a complete mathematical scheme for additive conjoint ordinal measurement over a much larger class of incompletely ordered relational structures than just those of the ?-transitive type An introduction to conjoint measurement begins the next Section and paves the way for presenting the arguments offered and the claims made by the scheme, the scheme itself, its scope and its limitations

SECTION 3

Additive Conjoint Measurement

3.1 CONVENTIONAL CONJOINT MEASUREMENT

At the end of the previous Section it was remarked that it is often possible to assume a greater amount of structure on the relational system than merely that which is explicitly stated by the ordering relation(s) The basis of this extra symmetry lies not in the definition of more ordering relations but rather in the patterns that can be discerned in the existing relations In other words, though the *type* of the structure (as defined in Section 1) is the same as that of a system that does not possess those symmetries, the additional symmetries allow us to model it in a more compact manner

The kind of internal symmetry that we seek is the property by which the set of alternatives may be expressed as the Cartesian product of several smaller sets, each called a *dimension* of the total structure

$$A = A^{n} \times A^{n-1} \times \quad \times A^{1}$$

Each dimension represents an *attribute* of the alternatives in $\mathcal{A}$ and is a

set of elements termed *attribute states* Thus, each alternative $a \in \mathcal{A}$ is expressible as an n-tuple - the particular set of attribute states $(a^n, \quad , a^1)$ where $a^i \in \mathcal{A}^i$ that determine it The weak ordering defined on $\mathcal{A}$ is very closely governed by the orderings on the attribute states a^i in each of the dimensions $\mathcal{A}^i$ but is not *fully* determined by them The exact feature that we demand of the relationship the orderings on the dimensions bear on the ordering on $\mathcal{A}$ is that it must permit the existence of an ordinal function f on $\mathcal{A}$ and ordinal functions f^i on $\mathcal{A}^i$ that are related through a composing function F by

$$f(a) = f(a^n, \quad , a^1) = F[f^n(a^n), f^{n-1}(a^{n-1}), \quad , f^1(a^1)]$$

satisfying

$$\begin{aligned} a_k \succ a_l \quad &\Leftrightarrow \quad f(a_k) > f(a_l) \\ &\Leftrightarrow \quad F[f^n(a_k{}^n), f^{n-1}(a_k{}^{n-1}), \quad , f^1(a_k{}^1)] > \\ &> \quad F[f^n(a_l{}^n), f^{n-1}(a_l{}^{n-1}), \quad , f^1(a_l{}^1)] \end{aligned} \qquad 3\ 1\text{-}1$$

where $a_k = (a_k{}^n, a_k{}^{n-1}, \quad , a_k{}^1)$ and $a_l = (a_l{}^n, a_l{}^{n-1}, \quad , a_l{}^1)$ The description given above is of the most general case of conjoint measurement However, the most adventurous models do not attempt anything more complex than the case of F being a polynomial in the f^i's, under which condition, the measurement scheme is termed *polynomial* conjoint measurement An important property that we require of any conjoint measurement scheme is that it be *decomposable* Decomposability permits us to decompose the utility on each $a \in \mathcal{A}$ into utilities on the individual $a^i \in \mathcal{A}^i$ This is possible in general only when the (so-called) *composing function*, F is one-one in each dimension

With this, we proceed to the most common (and simplest) form of conjoint measurement - *additive* conjoint measurement Here, we demand a

composing function that is no more complex than a mere summing function,

$$F[f^n(a^n), f^{n-1}(a^{n-1}), \ldots, f^1(a^1)] = \sum_{i=1}^{n} f^i(a^i)$$

Clearly, the composition is one-one in each dimension and is therefore decomposable

As an example, we may take the case of the requirements a student has to satisfy to get his M Tech degree The ordering imposed upon all postgraduates by their Cumulative Point Indices is an example of a weakly ordered system of alternatives that results from combining the particular states of a given set of attributes, namely, the set of courses completed by the student. The attribute states that correspond to the particular student are the *grades* he obtains in those courses The process of conjoint measurement consists of first assigning the ordinal functions for each dimension - here the *grade points* to each attribute state, the *grade* - and then composing the ordinal assignments to obtain the ordinal function on the total set (the *SPI*) by adding a weighted combination of the grade points

All of conjoint ordinal measurement is based on the presumption that not only is the total set a weak order, but further that each dimension is also weakly ordered. A set of axioms have been devised for additive conjoint measurement and an existence theorem [Luce & Tukey, 1964] is available that confirms the existence of a set of real functions satisfying 83 1-1 provided the system conforms to the axioms A less stringent set of axioms has been proposed for the particular case when the set of alternatives is finite (and, therefore, all the component dimensions are also finite) and a theorem demonstrating the existence of an additive conjoint scale given [Scott, 1964].

It is noticed that, upto the present, no attempt has been made to free additive conjoint ordinal measurement from the restriction imposed by the need for at least a weak order on the set of alternatives as well as on the set of states of every attribute Just as was argued in Section 2, it may be argued here, too, (in fact, even more forcefully than before) that such a stringent demand on the structural behaviour of any physical system can severely restrict the applicability of the model The aim of the present thesis is to present a model that admits of at least a particular form of incompleteness of the ordering relation so that we have a more general model for additive conjoint measurement than the conventional one Section 2 has already demonstrated the scheme for ordinal measurement for a *one dimensional* conjoint measurement problem (which is the undimensioned ordinal measurement problem) This Section will develop considerably on the already obtained results to define an axiomatic system of relational structures with well defined properties that lend themselves for additive conjoint measurement over multidimensional systems of alternatives for which each dimension is a ?-transitive incomplete structure The general approach of the following Subsections is to concentrate on developing the axiomatic scheme thoroughly and to say little of the measurement theoretic implications until this end is achieved to at least a satisfactory degree The various properties of our class of structures, which we call ***K-structures*** are discussed in the form of a series of theorems with proofs The final theorem included in this Section serves the purpose of showing that additive conjoint measurement is, indeed, possible when the incomplete systems dealt with conform to the requirements specified in the course of the development

3.2 K-STRUCTURES AND THE K-PRODUCT AXIOMS

The class of K-structures *is defined by the statements.*

- *all linearizable (? - transitive) structures are K - structures, and*
- *the K - product of two K - structures is also a K - structure*

K-PRODUCT AXIOMS

The *K-product,* denoted by $\circledast$, is an operation that assigns a K-structure $\mathcal{A} = \langle A, \succeq \rangle$ to every ordered pair of K-structures $\mathfrak{B} = \langle B, \succeq \rangle$ (the K-multiplier) and $\mathfrak{C} = \langle C, \succeq \rangle$ (the K-multiplicand) by the following axioms

$\mathcal{KP}1$ $A = B \times C$

Thus, A is just the Cartesian product of the sets in the operand structures Consequently, denoting A's cardinality by $\|A\| = \eta^A$,

$$\|A\| = \|B\| \cdot \|C\|, \qquad \text{or,} \quad \eta^A = \eta^B \cdot \eta^C$$

$\mathcal{KP}2$: $\forall \quad a_p = \langle b_i, c_k \rangle, \quad a_q = \langle b_j, c_l \rangle \in A,$

$$\forall \quad \circ \in \{\succeq, \preceq, ?\}$$

$$a_p \circ a_q \Leftrightarrow [b_i \circ b_j](1 - \delta_{i,j}) \vee [c_k \circ c_l](\delta_{i,j})$$

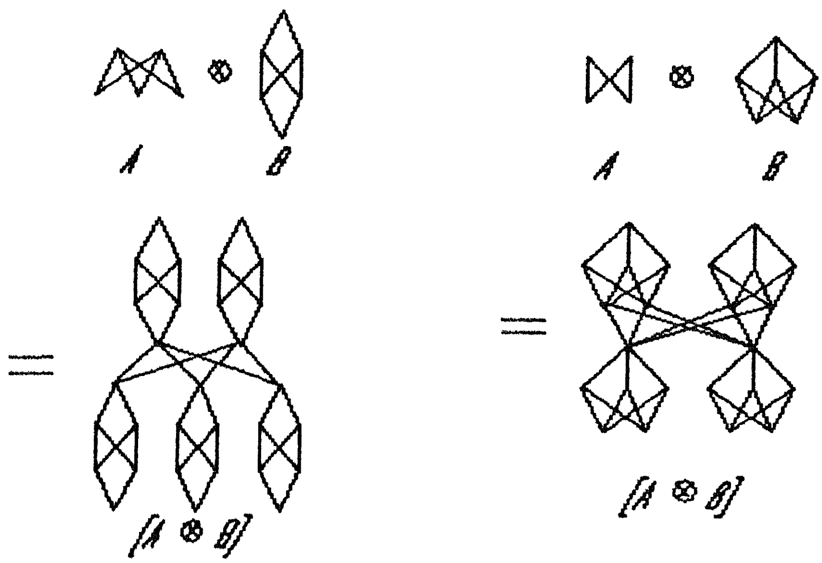
The notation adopted above is to be interpreted as follows We use the Kronecker delta as a symbol that has not the usual meaning of $\delta_{i,j} = 1$, $i = j$ and $\delta_{i,j} = 0$, $i \neq j$ but as a binary variable whose "value" is *true* when $i = j$, and whose "value" is *false* when $i \neq j$ Meanwhile, the symbol $(1 - \delta_{i,j})$ has the converse values under the respective conditions When these symbols are used in a logical expression such as $\mathcal{KP}2$ above, it

must be understood from the usage that either of the delta variables $\delta_{i,j}$ and $(1-\delta_{i,j})$ serves to silence the logical term to which it is attached whenever it (the delta variable) assumes the *false* state and, when it assumes the *true* state, it gates the term into the expression Though this notation is not conventional, drawing, as it does, ideas from calculus, set theory, and logic, it has been found to preserve consistency and leads to no contradictions As the following extensive use of the notation will show, the delta variables may even be operated upon algebraically as Boolean variables

In any incomplete system, (possibly not antisymmetric) any two alternatives ,x and y can have one of only 3 possible relationships $x \succeq y$ or $y \succeq x$ (which we define as the relation $x \preceq y$) or, if neither of the above is true, $x \,?\, y$ The symbol $\circ$ used above is meant to stand for any of these three possible relations To put $\mathcal{KP}2$ in words, the relation between a_p and a_q follows that existing between b_i and b_j except when $i = j$, when it follows instead, the relation between c_k and c_l

DIMENSION OF A K-STRUCTURE

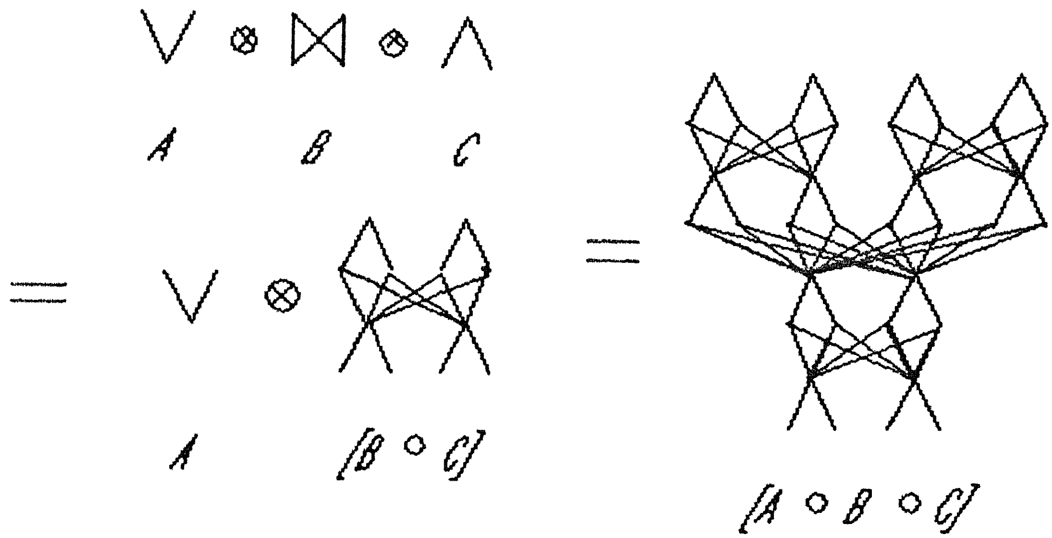
Following from the definition of a K-structure is the concept of the ***dimension*** of a K-structure The dimension of a K-structure is defined as the number of linearaizable structures that constitute it Thus, all linearizable structures are of dimension unity $\mathcal{F}$3 2-1 and $\mathcal{F}$3 2-2 show the Hasse diagrams of examples of two- and three dimensional K-structures respectively



STRUCTURE	VECTOR	|○|	||○||
A	[2 3]	2	5
B	[1 2 2 1]	4	6
A ○ B	—	8	30

STRUCTURE	VECTOR	|○|	||○||
A	[2 2]	2	4
B	[1 3 2]	3	6
A ○ B	--	6	24

Fig.3.2–1: 2 Dimensional K–structures

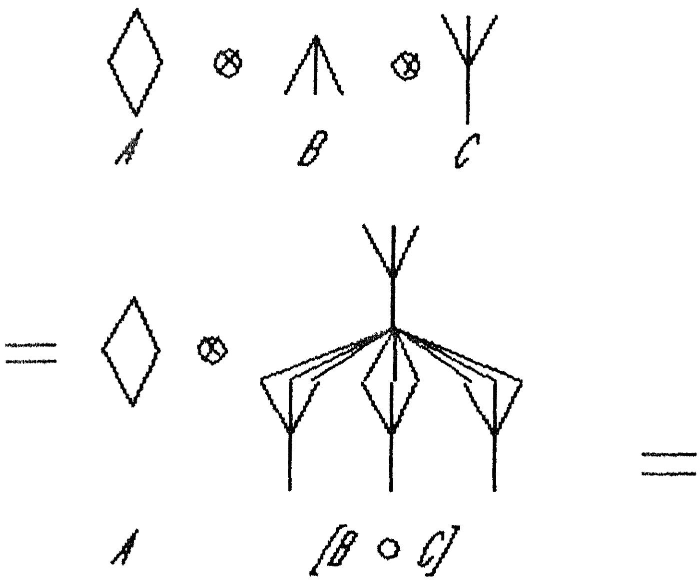


(a) Hasse Diagram

STRUCTURE	VECTOR	$\lvert\circ\rvert$	$\lVert\circ\rVert$
A	[2 1]	2	3
B	[2 2]	2	4
C	[1 2]	2	3
B ∘′ C	--	4	12
A ∘ B ∘ C	--	8	36

(b) Structural Details

Fig.3.2–2a: 3 Dimensional K-structure



STRUCTURE	VECTOR	$\lvert \circ \rvert$	$\lVert \circ \rVert$
A	[1 2 1]	3	4
B	[1 3]	2	4
C	[3 1 1]	3	5
B ∘ C	---	6	20
A ∘ B ∘ C	---	18	80

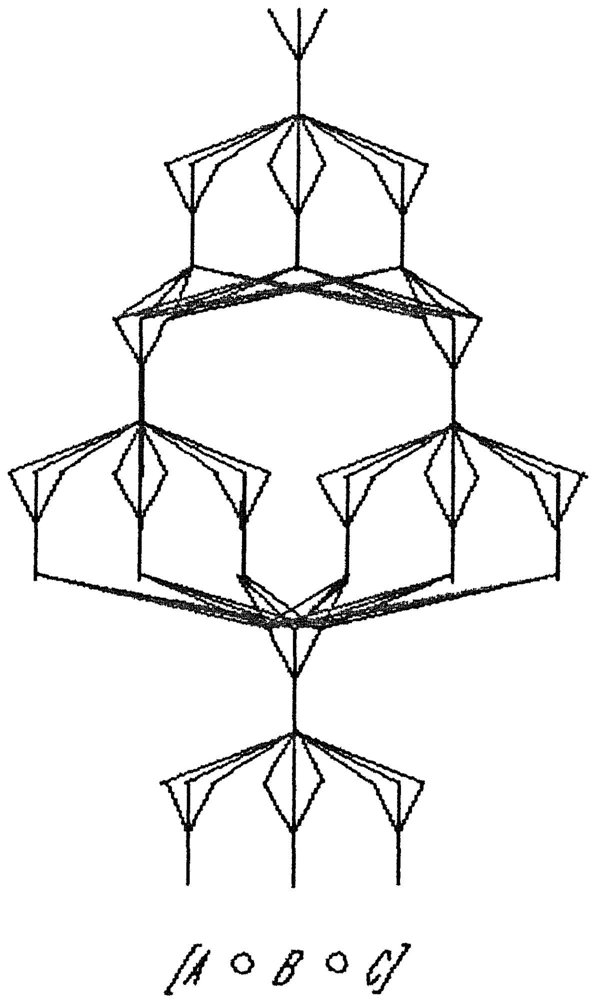


Fig.3.2–2b: 3 Dimensional K–structure

3.3 PROPERTIES OF THE K-PRODUCT AND K-STRUCTURES

We now present some theorems on the properties of the K-product operation and K-structures Proofs are provided for each theorem As the development proceeds, our knowledge of the peculiar behaviour of K-structures will widen, enabling us, finally, to determine the algebraic structure of the class of K-structures

THEOREM 3.3-1 NON - COMMUTATIVITY

The K - product operation is not commutative

PROOF: Consider $\mathcal{A} = \mathfrak{B} \otimes \mathfrak{C}$ and $\mathcal{A}' = \mathfrak{C} \otimes \mathfrak{B}$ From $\mathcal{KP}1$, we will have elements $a_p = (b_i, c_k)$, $a_q = (b_j, c_l) \in \mathcal{A}$, and correspondingly, $a'_p = (c_k, b_i)$, $a'_q = (c_l, b_j) \in \mathcal{A}'$ Already, we see that, though $\|A\| = \|A'\|$, the elements of each are different Further, from $\mathcal{KP}2$, we obtain the following expressions

$$a_p \circ a_q \Leftrightarrow [b_i \circ b_j](1 - \delta_{ij}) \vee [c_k \circ c_l](\delta_{ij})$$

$$a'_p \circ a'_q \Leftrightarrow [c_k \circ c_l](1 - \delta_{kl}) \vee [b_i \circ b_j](\delta_{kl})$$

We see that the difference between $(a_p \circ a_q)$ and $(a'_p \circ a'_q)$ is caused by the non-symmetric implication from $\mathcal{KP}2$ that $(a_p, \circ a_q)$ follows the multiplicand relation *only when the multiplier points are identical* and is independent of whether or not the *multiplicand* points are identical - thus treating multiplier and multiplicand differently While in $\mathcal{A}$, for $i \neq j$,

$$a_p \circ a_q \Leftrightarrow b_i \circ b_j,$$

in $\mathcal{A}'$, whenever $k \neq l$, and quite independently of whether or not $i \neq j$,

$$a'_p \circ a'_q \Leftrightarrow c_k \circ c_l$$

Clearly, in the general case, the relation $(b_i \circ b_j)$ need not be the same as $(c_k \circ c_l)$ and so, $(a_p \circ a_q) \neq (a'_p \circ a'_q)$, hence $\otimes$ is not commutative □

THEOREM 3.3-2 ASSOCIATIVITY

The K - product operation is associative

PROOF: Consider $\mathcal{A} = \mathfrak{B} \otimes (\mathfrak{C} \otimes \mathfrak{D})$ and $\mathcal{A}' = (\mathfrak{B} \otimes \mathfrak{C}) \otimes \mathfrak{D}$ We need to prove that $\mathcal{A}$ and $\mathcal{A}'$ are isomorphic structures We do so by expanding each side separately using the axioms and demonstrating that $\mathcal{A} = \mathcal{A}'$ To begin with, the cardinalities of both can be shown to be the same,

$$\|A\| = \|B\| \cdot \|(C \times D)\| = \|B\| \cdot \|C\| \cdot \|D\|$$
$$= \|(B \times C)\| \cdot \|D\| = \|A'\|$$

We temporarily name the intermediate structures $\mathfrak{E} = \mathfrak{C} \otimes \mathfrak{D}$ and $\mathfrak{F} = \mathfrak{B} \otimes \mathfrak{C}$, thus, $\mathcal{A} = \mathfrak{B} \otimes \mathfrak{E}$ and $\mathcal{A}' = \mathfrak{F} \otimes \mathfrak{D}$ For $e_f, e_g \in E$, $d_m, d_n \in D$, we have from $\mathfrak{KP}2$ that

$$e_f \circ e_g \Leftrightarrow [c_k \circ c_l](1 - \delta_{kl}) \quad \vee \quad [d_m \circ d_n](\delta_{kl})$$

and thus, for $a_p, a_q \in \mathcal{A}$,

$$a_p \circ a_q \Leftrightarrow [b_i \circ b_j](1 - \delta_{ij}) \quad \vee \quad [e_f \circ e_g](\delta_{ij})$$

Substituting for $e_f \circ e_g$ in the above expression,

$$\begin{aligned} a_p \circ a_q \Leftrightarrow \; & [b_i \circ b_j](1 - \delta_{ij}) \\ & \vee \quad [c_k \circ c_l](1 - \delta_{kl})(\delta_{ij}) \\ & \vee \quad [d_m \circ d_n](\delta_{kl})(\delta_{ij}) \end{aligned} \qquad 3\,3\text{-}2a$$

On the other hand, for $f_r, f_s \in \mathfrak{F}$, $\mathfrak{KP}2$ yields

$$f_r \circ f_s \Leftrightarrow [b_i \circ b_j](1 - \delta_{ij}) \quad \vee \quad [c_k \circ c_l](\delta_{ij})$$

which, in turn, implies that, for $a'_p, a'_q \in \mathcal{A}'$,

$$a'_p \circ a'_q \Leftrightarrow [f_r \circ f_s](1-\delta_{rs}) \vee [d_m \circ d_n](\delta_{rs})$$

Substituting for $f_r \circ f_s$ in the above expression, we obtain

$$a'_p \circ a'_q \Leftrightarrow [b_i \circ b_j](1-\delta_{i,j})(1-\delta_{rs})$$
$$\vee [c_k \circ c_l](\delta_{i,j})(1-\delta_{rs}) \vee [d_m \circ d_n](\delta_{rs})$$

Since $\delta_{rs} = \delta_{i,j}\cdot\delta_{kl}$, the above becomes

$$a'_p \circ a'_q \Leftrightarrow [b_i \circ b_j](1-\delta_{i,j})(1-\delta_{i,j}\cdot\delta_{kl})$$
$$\vee [c_k \circ c_l](\delta_{i,j})(1-\delta_{i,j}\cdot\delta_{kl})$$
$$\vee [d_m \circ d_n](\delta_{i,j}\cdot\delta_{kl}) \qquad 3.3\text{-}2b$$

Finally, we note that since the following equalities are shown to be valid

- $(1-\delta_{i,j})(1-\delta_{i,j}\cdot\delta_{kl}) = 1-\delta_{i,j}-\delta_{i,j}\cdot\delta_{kl}+\delta_{i,j}\cdot\delta_{i,j}\cdot\delta_{kl}$

  $= (1-\delta_{i,j})$ (because $\delta_{i,j}\cdot\delta_{i,j} = \delta_{i,j}$)

- $(\delta_{i,j})(1-\delta_{i,j}\cdot\delta_{kl}) = \delta_{i,j}-\delta_{i,j}\cdot\delta_{i,j}\cdot\delta_{kl} = \delta_{i,j}-\delta_{i,j}\cdot\delta_{kl}$

  $= (1-\delta_{kl})(\delta_{i,j})$

we may use them in §3.3-2b Now that §3.3-2a and §3.3-2b are identical, we are assured that $a'_p \circ a'_q = a_p \circ a_q$ holds for all p and q □

THEOREM 3.3-3 INDEPENDENCE

The K - product ensures the independence of the operand structures where independence is interpreted as the following property:

$$b_i \circ b_j \Rightarrow (b_i, c_k) \circ (b_j, c_k), \quad \forall c_k \in \mathfrak{C} \qquad 3.3\text{-}3a$$

$$c_k \circ c_l \Rightarrow (b_i, c_k) \circ (b_i, c_l), \quad \forall b_i \in \mathfrak{B} \qquad 3.3\text{-}3b$$

PROOF: Let $\mathcal{A} = \mathfrak{B} \otimes \mathfrak{C}$ and let $a_p = (b_i, c_k)$, $a_q = (b_j, c_l)$. Then, taking $k = l$ in $\mathfrak{KP}2$, we get §3.3-3a and taking $i = j$, we get §3.3-3b. This clearly proves the independence property Indeed, independence has been the among the primary considerations in the formulation of the axioms □

THEOREM 3.3-4 NON-LINEARIZABILITY

The K - product is generally non-linearizable, a sufficient condition being the non-emptiness of ? in the K - multiplier.

PROOF: Suppose $\exists$ $b_i, b_j \in \mathfrak{B}$ s t $b_i \,?\, b_j$ Then, by $\mathfrak{KP}$2,

$$(b_i, \ c_k) \,?\, (b_j, \ c_l) \qquad\qquad \forall \ c_k, c_l \ \in \ \mathfrak{C}$$

Now, if it happens that, for some particular $c_k, c_l \in \mathfrak{C}$, $c_k \succsim c_l$, we would arrive at the following expressions upon applying $\mathfrak{KP}$2 and $\mathfrak{T}$3 3-3

- $(b_i, \ c_k) \,?\, (b_j, \ c_l)$
- $(b_i, \ c_l) \,?\, (b_j, \ c_l)$
- $(b_i, \ c_k) \succsim (b_i, \ c_l)$

The above set of expressions violate ?-transitivity From $\mathfrak{T}$2 3-1, it follows that the product is not linearizable □

THEOREM 3.3-5 DIMENSIONALITY

The dimension of the K - product is the sum of the dimensions of the operands.

PROOF: Let $\mathfrak{B}^{(m)}$ and $\mathfrak{C}^{(n)}$ be operands of dimension m and n respectively Then, by the definition of the dimension of a K-structure, there exist ordered ($\mathfrak{T}$3.3-1) sequences of one-dimensional linearizable K-structures such that

$$\mathfrak{B}^{(m)} = \mathfrak{B}^{m} \otimes \mathfrak{B}^{m-1} \otimes \qquad \otimes \mathfrak{B}^{1}$$

$$\mathfrak{C}^{(n)} = \mathfrak{C}^{n} \otimes \mathfrak{C}^{n-1} \otimes \qquad \otimes \mathfrak{C}^{1}$$

The K-product of $\mathfrak{B}^{(m)}$ and $\mathfrak{C}^{(n)}$ would be

$$\mathfrak{B}^{(m)} \otimes \mathfrak{C}^{(n)} = (\mathfrak{B}^{m} \otimes . \qquad \otimes \mathfrak{B}^{1}) \otimes (\mathfrak{C}^{n} \otimes \qquad \otimes \mathfrak{C}^{1})$$

By $\mathfrak{T}$3.3-2, the above is equal to

$$\mathfrak{B}^{m} \otimes \ldots \otimes \mathfrak{B}^{1} \otimes \mathfrak{C}^{n} \otimes \qquad \otimes \mathfrak{C}^{1}$$

which is a K-product sequence of $m + n$ linearizable structures and therefore a K-structure of dimension $m + n$ $\square$

THEOREM 3.3-6 IDENTITY

The (left and right) identity elements for the class of K - structures coincide and are the zero - dimensional structure $\mathcal{E}$ *with* $E = \{e\}$ *(hence,* $\|\mathcal{E}\| = 1$*),* $\succsim \; = \{(e,e)\}$ *(or* $\succ \; = \emptyset$*) and* $? = \emptyset$

PROOF (Left Identity): Consider $\mathcal{S} = \mathcal{E} \otimes \mathcal{A}$, by $\mathcal{KP}1$

$$\|\mathcal{S}\| = \|\mathcal{E} \otimes \mathcal{A}\| = \|\mathcal{E}\| \cdot \|\mathcal{A}\| = \|\mathcal{A}\|$$

As $\|\mathcal{E}\| = 1$, if $e_i, e_j \in \mathcal{E}$, then $e_i \equiv e_j$ Therefore, $\delta_{ij} = 1$ and by $\mathcal{KP}2$

$$s_p \circ s_q \Leftrightarrow a_k \circ a_l$$

Since this holds for $\forall\, p, q \in \mathcal{S}$ and $\forall\, k, l \in \mathcal{A}$ they must be isomorphic

PROOF (Right Identity) We take $\mathcal{S}' = \mathcal{A} \otimes \mathcal{E}$, again, by $\mathcal{KP}1$

$$\|\mathcal{S}'\| = \|\mathcal{A} \otimes \mathcal{E}\| = \|\mathcal{A}\| \cdot \|\mathcal{E}\| = \|\mathcal{A}\|$$

This time, by $\mathcal{KP}2$,

$$s_p \circ s_q \Leftrightarrow [a_i \circ a_j](1 - \delta_{ij}) \quad \vee \quad [e_k \circ e_l](\delta_{ij})$$

When $i = j$, meaning that $a_i \equiv a_j$,

$$s_p \circ s_q \Leftrightarrow e_k \circ e_l$$

Since $e_k \equiv e_l$, $s_p \equiv s_q$, whenever $i \neq j$,

$$s_p \circ s_q \Leftrightarrow a_i \circ a_j$$

This means that, again, $\mathcal{S}'$ and $\mathcal{A}$ are isomorphic

Finally, as $\mathcal{A} \otimes \mathcal{E} = \mathcal{A}$, by $\mathcal{T}$3 3-5, the dimension of $\mathcal{E}$ must be zero. $\square$

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3.4 LEVELS OF A K-STRUCTURE

The set of **levels** $\{\ell_n\}$, of a K-structure is an ordered set of subsets of $\mathcal{A}$ and is defined as that which is obtained by the application of the following algorithm

ALGORITHM 3.4-1 LEVEL GENERATION

$\alpha_1 \triangleq \mathcal{A}$

repeat

$\ell_n \triangleq \{ x \mid (x \in \alpha_n) \wedge (\sim\exists\ y \in \alpha_n \mid y \succeq x)\}$

$\alpha_{n+1} \triangleq \alpha_n \cap \overline{\ell_n}$

if $|\alpha_{n+1}| > 0$, repeat

end.

The levels of a K-structure will be seen to play a central role in our analysis of the K-product operation and the K-structures themselves In fact, it will follow from our discussions that all elements ${}_n a_i \in \ell_n$ will be assigned the same ordinal number in our measurement procedure for K-structures. Formally, this would mean

$$f({}_n a_i) = f({}_n a_j) \qquad \forall\ {}_n a_i,\ {}_n a_j \in \ell_n$$

Thus, the ℓ_n have to be equivalence classes For all future use, we adopt the following notation the number of elements of $\mathcal{A}$ in $\ell_n = |\ell_n|$, the number of levels in $\mathcal{A} = |\mathcal{A}| = \lambda$ In this Section, we explore the properties of the levels of a K-structure through a set of theorems that relate them to the already explored properties of K-structures

THEOREM 3.4-1 LEVEL DISJOINTNESS

The levels form a partition of $\mathcal{A}$

PROOF: We observe that $\ell_n \subseteq \mathcal{A}$ Moreover,

$$\alpha_{n+1} \triangleq \alpha_n \cap \overline{\ell_n} \Rightarrow \sim\exists \ x \ \text{s t} \ (x \in \alpha_{n+1}) \wedge (x \in \ell_n)$$

Therefore,

$$\ell_n \cap \ell_m = \emptyset \qquad\qquad m \neq n,$$

$$\alpha_{n+1} \cap \ell_n = \emptyset$$

As $\alpha_n = \alpha_{n+1} \cup \ell_n$, and, as, by the definition of a level in the defining algorithm, it is necessary that $|\ell_n| > 0$, we will have $|\alpha_n| > |\alpha_{n+1}|$ Finally, as $\mathcal{A}$ is assumed to be finite,

$$\exists \ \lambda \ \text{s t} \ \alpha_{\lambda+1} = \emptyset$$

which is the point at which the level generating algorithm terminates Thus, from the above discussion, it follows that, for any α_n, we may write,

$$A = \left(\bigcup_{i=1}^{n-1} \ell_i\right) \cup \alpha_n \qquad \text{and} \qquad A = \bigcup_{i=1}^{\lambda} \ell_i$$

Finally, from disjointness of the levels, we get the additional result,

$$\|\mathcal{A}\| \triangleq \eta = \sum_{n=1}^{\lambda^A} |\ell_n| \qquad \square$$

THEOREM 3.4-2 PURE INDETERMINANCY

The restriction of the (strict) ordering relation is necessarily empty in each level.

PROOF: Assuming the opposite directly points to a contradiction From the definition of a level

$$\ell_n \triangleq \{ x \mid (x \in \alpha_n) \wedge \sim(\exists \ y \in \alpha_n \mid y \succeq x)\},$$

it is a necessary condition that *each* ${}_n a_i \in \ell_n$ be a maximum of α_n. Suppose

$$\exists \; {}_n a_i, {}_n a_j \in \ell_n \quad s.t. \quad {}_n a_i \succeq {}_n a_j$$

If ${}_n a_i \succeq {}_n a_j$, then, clearly, ${}_n a_j$ is not a maximum of α_n. □

THEOREM 3.4-3 LEVEL DISTINCTNESS

All maximal chains in a K - structure are of the same length equal to the number of levels in it.

PROOF: We begin by first proving only that the K-product operation is distinctness preserving - meaning that, if the K-operands are level distinct, then their K-product preserves the property. Moreover, it is necessary to prove even this only for the particular case when the K-multiplier is a linearizable structure rather than a general K-structure. The reason is that, *any* K-structure, by definition, is ultimately the multiple K-product of linearizables. The proof of the theorem will follow by induction from

- the level distinctness preserving property of the K-product
- the inherent level distinctness of linearizable structures (§2.3-1)

We adopt the following notation henceforth: the K-structure that is the product is called $\mathcal{A}^{(n)}$, the K-multiplier, $\mathcal{A}^{n}$, and the K-multiplicand, $\mathcal{A}^{(n-1)}$.

$$\mathcal{A}^{(n)} = \mathcal{A}^{n} \otimes \mathcal{A}^{(n-1)}$$

$\mathcal{A}^{(n)}$ is assumed to be the K-product of n linearizables (is of dimension n), $\mathcal{A}^{n}$, is (termed) the nth dimension and is the nth linearizable member of the product and finally, $\mathcal{A}^{(n-1)}$ (dimension $n-1$), the K-product of the remaining $n-1$ members. In other words, if we write, in the same fashion,

$$\mathcal{A}^{(n-1)} = \mathcal{A}^{n-1} \otimes \mathcal{A}^{(n-2)}$$

and so on, we get finally,

$$\mathcal{A}^{(2)} = \mathcal{A}^{2} \otimes \mathcal{A}^{(1)} \qquad \text{where} \qquad \mathcal{A}^{(1)} \triangleq \mathcal{A}^{1}$$

In general, therefore, the $\mathcal{A}^{i}$'s are, respectively, the ith linearizable components in $\mathcal{A}^{(n)}$ and the $\mathcal{A}^{(i)}$'s are, respectively, the K-products of the first i component linearizable structures A particular element of any linearizable structure $\mathcal{A}^{i}$ are denoted by a_k^{i}, and any general element by a^{i} Likewise, $a_k^{(i)}, a^{(i)} \in \mathcal{A}^{(i)}$ A representation of the kind $a^{(n)} = (a^{n}, a^{(n-1)})$, which leads to an n-tuple representation for $a^{(n)}$ of the form $(a^{n}, a^{n-1}, \quad , a^{1})$ directly follows

$$a^{(n)} = (a^{n}, a^{(n-1)}) = (a^{n}, a^{n-1}, a^{(n-2)}) = \qquad =$$
$$= (a^{n}, \quad , a^{1})$$

Under this notation, the K-product axioms would be as follows

$\mathcal{KP}1$ $\quad A^{(n)} = A^{n} \times A^{(n-1)}$

which would imply that

$$\|A^{(n)}\| = \|A^{n}\| \cdot \|A^{(n-1)}\|, \qquad \text{or,}$$
$$\eta^{(n)} = \eta^{n} \cdot \eta^{(n-1)}$$

$\mathcal{KP}2\cdot$ $\quad \forall \quad a_p^{(n)} = (a_i^{n}, a_k^{(n-1)}), \quad a_q^{(n)} = (a_j^{n}, a_l^{(n-1)}) \in A^{(n)},$

$$\forall \quad \circ \in \{\succeq, \preceq, ?\}$$

$$a_p^{(n)} \circ a_q^{(n)} \Leftrightarrow [a_i^{n} \circ a_j^{n}](1 - \delta_{ij}) \vee [a_k^{(n-1)} \circ a_l^{(n-1)}](\delta_{ij})$$

Further,

$$|\mathcal{A}^{(n)}| \triangleq \lambda^{(n)}, \qquad \|\mathcal{A}^{(n)}\| \triangleq \eta^{(n)},$$
$$|\mathcal{A}^{n}| \triangleq \lambda^{n}, \qquad \|\mathcal{A}^{n}\| \triangleq \eta^{n}, \qquad \textit{and}$$
$$|\mathcal{A}^{(n-1)}| \triangleq \lambda^{(n-1)}, \qquad \|\mathcal{A}^{(n-1)}\| \triangleq \eta^{(n-1)}$$

Finally, the gth level of $\mathcal{A}^{(n)} \triangleq \ell_g^{(n)}$, of $\mathcal{A}^{n} \triangleq \ell_g^{n}$ and of $\mathcal{A}^{(n-1)} \triangleq \ell_g^{(n-1)}$

We are now at last ready to begin the actual proof As $\mathcal{A}^{(n-1)}$ is *assumed* to be level distinct, we know, by the definition of level distinctness,

that all maximal chains in it will be of the same length $\lambda^{(n-1)}$ Let the $\lambda^{(n-1)}$ levels of $\mathcal{A}^{(n-1)}$ be named, respectively, $\ell_1^{(n-1)}, \ell_2^{(n-1)}, \ldots, \ell_{\lambda(n-1)}^{(n-1)}$ If we choose $a_k^{(n-1)} \in \ell_1^{(n-1)}$ and $a_l^{(n-1)} \in \ell_{\lambda(n-1)}^{(n-1)}$, then these two points are a maximum and a minimum, respectively, in $\mathcal{A}^{(n-1)}$, and they are, therefore, the end points of one or more maximal chains in $\mathcal{A}^{(n-1)}$ From $\mathcal{HP}2$, for $i = j$,

$$a_p^{(n)} \circ a_q^{(n)} \Leftrightarrow a_k^{(n-1)} \circ a_l^{(n-1)}$$

If we define the *local structure* of $\mathcal{A}^{(n)}$ as the restriction of $\mathcal{A}^{(n)}$ obtained by restricting $\mathcal{A}^n$ to just the single point a_i^n, then the above expression means that the local structure of $\mathcal{A}^{(n)}$ is isomorphic to $\mathcal{A}^{(n-1)}$ This, in fact, follows from §3 3-6, as our restriction of $\mathcal{A}^n$ to a single point amounts to reducing $\mathcal{A}^n$ to (an isomorphism of) the identity element The local structure is clearly quite independent of which particular a_i^n is selected Let us denote this local substructure of $\mathcal{A}^{(n)}$ by ${}^i\mathcal{A}^{(n)}$ Choosing another point $a_j^n \in \mathcal{A}^n$ such that $a_i^n \succ a_j^n$, and considering the local substructure of $\mathcal{A}^{(n)}$ about it, namely, ${}^j\mathcal{A}^{(n)}$, we get from $\mathcal{HP}2$ that

$${}_ia^{(n)} \succ {}_ja^{(n)} \qquad \forall\, {}_ia^{(n)} \in {}^i\mathcal{A}^{(n)},\ {}_ja^{(n)} \in {}^j\mathcal{A}^{(n)}$$

If we choose

$$a_i^n,\ a_j^n \in \mathcal{A}^n \quad \text{s t} \quad \sim\exists\ a^n \in \mathcal{A}^n \text{ satisfying } a_i^n \succ a^n \succ a_j^n$$

then, again from $\mathcal{HP}2$, it follows that

$$\sim\exists\ a^{(n)} \notin {}^i\mathcal{A}^{(n)} \cup {}^j\mathcal{A}^{(n)} \quad \text{s t} \quad {}_ia^{(n)} \succ a^{(n)} \succ {}_ja^{(n)}$$

Under this condition, the substructure ${}^i\mathcal{A}^{(n)} \cup {}^j\mathcal{A}^{(n)}$ will clearly be level distinct and therefore, will have all its maximal chains of length equal to

$$\left|{}^i\mathcal{A}^{(n)} \cup {}^j\mathcal{A}^{(n)}\right| = \left|{}^i\mathcal{A}^{(n)}\right| + \left|{}^j\mathcal{A}^{(n)}\right|$$
$$= 2\cdot\left|\mathcal{A}^{(n-1)}\right| = 2\cdot\lambda^{(n-1)}$$

The maxima of ${}^i\mathcal{A}^{(n)} \cup {}^j\mathcal{A}^{(n)}$ will all be points $a_p^{(n)} = (a_i^n, {}_1a_k^{(n-1)})$ such that the $a_k^{(n-1)}$ are maxima of $\mathcal{A}^{(n-1)}$ (because ${}_1a_k^{(n-1)} \in \ell_1^{(n-1)}$) The minima, on the

other hand, will all be points $a_q^{(n)} = (a_v^{\,n},\ {}_{\lambda(n-1)}a_l^{(n-1)})$ (as ${}_{\lambda(n-1)}a_l^{(n-1)} \in \ell_{\lambda(n-1)}^{(n-1)}$.

We now extrapolate the construction detailed above Using ${}_1a^n$, ${}_2a^n$, . . , ${}_{\lambda n}a^n$, which together constitute a maximal chain in $\mathcal{A}^n$, we construct a *maximal substructure chain* in $\mathcal{A}^{(n)}$,

$${}^1\mathcal{A}^{(n)} \cup {}^2\mathcal{A}^{(n)} \cup \quad \cup {}^{\lambda n}\mathcal{A}^{(n)}$$

This, again, by the same argument as before, is level distinct Therefore, all maximal chains in it will be of length equal to the number of levels in it

$$\left|{}^1\mathcal{A}^{(n)}\right| + \left|{}^2\mathcal{A}^{(n)}\right| + \quad + \left|{}^{\lambda n}\mathcal{A}^{(n)}\right|$$
$$= \lambda^n \cdot \lambda^{(n-1)}$$

Since all maximal substructure chains are isomorphic, as is clear from

- the fact that all maximal chains in $\mathcal{A}^n$ are of length λ^n
- the fact that the substructures are all isomorphic to $\mathcal{A}^{(n-1)}$ and hence, to one another

we can conclude that *all* maximal chains in $\mathcal{A}^{(n)}$ are of equal length

$$\lambda^n \cdot \lambda^{(n-1)} \triangleq \lambda^{(n)}$$

Thus, $\mathcal{A}^{(n)}$ is also a level distinct structure with

$$\left|\mathcal{A}^{(n)}\right| = \left|\mathcal{A}^n\right| \cdot \left|\mathcal{A}^{(n-1)}\right|$$

As has already been stated in the beginning of the proof, the level distinctness of a K-structure follows from this by induction □

THEOREM 3.4-4 LEVEL CARDINALITY

$$\left|\ell_\gamma^{(n)}\right| = \left|\ell_\alpha^{\,n}\right| \cdot \left|\ell_\beta^{(n-1)}\right| \qquad \textit{where}$$
$$\gamma = (\alpha-1)\cdot\lambda^{(n-1)} + \beta \qquad \textit{or, equivalently, where}$$
$$\alpha = \lceil \gamma/\lambda^{(n-1)} \rceil \qquad \textit{and} \quad \beta = (\gamma) \bmod \lambda^{(n-1)}.$$

PROOF: As is already known the substructures are isomorphic to $\mathcal{A}^{(n-1)}$ Because every $_{\alpha}a_{\iota}^{n} \in \ell_{\alpha}^{n}$, every $^{\iota}\mathcal{A}^{(n)}$ will be the αth in some maximal substructure chain of $\mathcal{A}^{(n)}$ Consequently, there will be $\alpha-1$ substructures $^{h}\mathcal{A}^{(n)}$ with the property

$$^{h}a^{(n)} \succ {}^{\iota}a^{(n)} \qquad \forall\ ^{h}a^{(n)} \in {}^{h}\mathcal{A}^{(n)},\ ^{\iota}a^{(n)} \in {}^{\iota}\mathcal{A}^{(n)}$$

Since each substructure contributes $\lambda^{(n-1)}$ levels to the maximal substructure chain, the $\alpha-1$ substructures $^{h}\mathcal{A}^{(n)}$ will together contribute $(\alpha-1)\cdot\lambda^{(n-1)}$ levels This means that $(\alpha-1)\cdot\lambda^{(n-1)}$ levels precede the first level in $^{\iota}\mathcal{A}^{(n)}$ in order of appearance when the level generating algorithm is run, or, that the ordinal number of the first level in $^{\iota}\mathcal{A}^{(n)}$ is $(\alpha-1)\cdot\lambda^{(n-1)} + 1$ in the maximal substructure chain More generally, the βth level in $^{\iota}\mathcal{A}^{(n)}$ is of ordinal number $(\alpha-1)\cdot\lambda^{(n-1)} + \beta$ in the maximal substructure chain From the isomorphicity of all maximal substructure chains, it is clear that this level must indeed be the γth in $\mathcal{A}^{(n)}$ As $\beta \leq \lambda^{(n-1)}$, we conclude that $\beta = (\gamma) \bmod \lambda^{(n-1)}$ Again, because $\beta \leq \lambda^{(n-1)}$, $\alpha-1 \leq \gamma/\lambda^{(n-1)} \leq \alpha$ which is the same as saying that $\alpha = \lceil \gamma/\lambda^{(n-1)} \rceil$

Consider $_{\alpha}a_{\iota}^{n} \in \ell_{\alpha}^{n} \subset \mathcal{A}^{n}$ and $_{\beta}a_{k}^{(n-1)} \in \ell_{\beta}^{(n-1)} \subset \mathcal{A}^{(n-1)}$ By $\mathfrak{T}$3 4-2,

$$_{\alpha}a_{\iota}^{n} \supset {}_{\alpha}a_{\nu}^{n} \qquad \forall\ _{\alpha}a_{\iota}^{n},\ _{\alpha}a_{\nu}^{n} \in \ell_{\alpha}^{n}$$

Therefore, by $\mathfrak{HP}$2,

$$^{\iota}a^{(n)} \supset {}^{\nu}a^{(n)} \qquad \forall\ ^{\iota}a^{(n)} \in {}^{\iota}\mathcal{A}^{(n)},\ ^{\nu}a^{(n)} \in {}^{\nu}\mathcal{A}^{(n)}$$

There will be $|\ell_{\alpha}^{n}|$ such substructures $^{\iota}\mathcal{A}^{(n)}$, the set of which we shall denote by $\{^{\iota}\mathcal{A}^{(n)}\}$ Moreover, each $^{\iota}\mathcal{A}^{(n)}$ has within it, $|\ell_{\beta}^{(n-1)}|$ points at its βth level Therefore,

$$|\ell_{\gamma}^{(n)}| = |\ell_{\alpha}^{n}| \cdot |\ell_{\beta}^{(n-1)}| \qquad \square$$

Thus, from $\eta^{(n)} = \eta^{n} \cdot \eta^{(n-1)}$ (a consequence of $\mathfrak{HP}$1), $\mathfrak{T}$3 4-1

(disjointness) and the above theorem on level cardinality, we get

$$\sum_{\gamma=1}^{\lambda^{(n)}} \left|\ell_\gamma^{(n)}\right| = \left(\sum_{\alpha=1}^{\lambda^n} \left|\ell_\alpha^{\,n}\right|\right) \cdot \left(\sum_{\beta=1}^{\lambda^{(n-1)}} \left|\ell_\beta^{(n-1)}\right|\right)$$

3.5 THE MORE SIGNIFICANT RESULTS

It is an indisputable fact that a knowledge of the class of automorphisms of a newly defined type of structure is of considerable value in understanding its properties and in locating it in the existing edifice of the theory. ¶3.5-1 defines the class of automorphisms of a K-structure from which it is seen that the class grows in size rapidly as the number of dimensions increases and is in fact the product of the numbers of automorphisms of the component dimensions.

THEOREM 3.5-1 AUTOMORPHISMS

The automorphisms of a K - structure are the K - products of the automorphisms of their component structures.

PROOF: Let

$$\mathcal{A}^{(n)} = \mathcal{A}^{n} \otimes \mathcal{A}^{(n-1)}$$

We provide a recursive proof. As $\mathcal{A}^{n}$ is a linearizable structure, we already know about its class of automorphisms (¶2.3-2). To begin with, we prove that *if* we *assume* that $\mathcal{A}^{*(n-1)}$ is isomorphic to $\mathcal{A}^{(n-1)}$, and *choose* an $\mathcal{A}^{*n}$ isomorphic to $\mathcal{A}^{n}$, *then*

$$\mathcal{A}^{*(n)} = \mathcal{A}^{*n} \otimes \mathcal{A}^{*(n-1)}$$

is isomorphic to $\mathcal{A}^{(n)}$ As the above statement is meant to apply for all n, we are already assured, since the $\mathcal{A}^{*n}$ are *chosen* (with the help of ℱ2 3-2) to be isomorphic to $\mathcal{A}^{n}$, that $\mathcal{A}^{*(n)}$ is undoubtedly a K-structure Because $\mathcal{A}^{*n}$ is isomorphic to $\mathcal{A}^{n}$,

$$a_i^{\,n} \circ a_j^{\,n} \quad \Rightarrow \quad a_i^{\,*n} \circ a_j^{\,*n} \qquad 3\,5\text{-}1a$$

Likewise, for $\mathcal{A}^{(n-1)}$ and $\mathcal{A}^{*(n-1)}$, by assumption,

$$a_k^{\,(n-1)} \circ a_l^{\,(n-1)} \Leftrightarrow a_k^{\,*(n-1)} \circ a_l^{\,*(n-1)} \qquad 3\,5\text{-}1b$$

We recall that

$$a_p^{\,(n)} = (a_i^{\,n},\ a_k^{\,(n-1)}) \quad \text{and} \quad a_q^{\,*(n)} = (a_j^{\,*n},\ a_l^{\,*(n-1)})$$

From 𝒦𝒫2,

$$a_p^{\,(n)} \circ a_q^{\,(n)} \Leftrightarrow [a_i^{\,n} \circ a_j^{\,n}](1-\delta_{ij}) \ \vee\ [a_k^{\,(n-1)} \circ a_l^{\,(n-1)}](\delta_{ij})$$

Substituting 83 4-1 in the above, we get

$$a_p^{\,(n)} \circ a_q^{\,(n)} \Leftrightarrow [a_i^{\,*n} \circ a_j^{\,*n}](1-\delta_{ij})$$
$$\vee\ [a_k^{\,*(n-1)} \circ a_l^{\,*(n-1)}](\delta_{ij})$$

As it has already been argued that $\mathcal{A}^{*(n)}$ cannot but be a K-structure, 𝒦𝒫2 applies for it as well,

$$a_p^{\,*(n)} \circ a_q^{\,*(n)} \Leftrightarrow [a_i^{\,*n} \circ a_j^{\,*n}](1-\delta_{ij})$$
$$\vee\ [a_k^{\,*(n-1)} \circ a_l^{\,*(n-1)}](\delta_{ij})$$

Isomorphicity of $\mathcal{A}^{(n)}$ and $\mathcal{A}^{*(n)}$ follows from the two expressions above

$$a_p^{\,(n)} \circ a_q^{\,(n)} \Leftrightarrow a_p^{\,*(n)} \circ a_q^{\,*(n)} \qquad \square$$

The next theorem included in this Subsection is the inevitable representation theorem for additive conjoint measurement over the class of K-structures A representation theorem is the first that has to be proved whenever a new measurement scheme is proposed, but in our case, it had to wait until all the prerequisite explanations had been given

THEOREM 3.5-2 REPRESENTATION

A scale

$$a_p^{(n)} \succ a_q^{(n)} \Rightarrow f(a_p^{(n)}) > f(a_q^{(n)}) \qquad \forall \; a_p^{(n)}, a_q^{(n)} \in \mathcal{A}^{(n)}$$

that satisfies

$$f^{(n)}(a_r^{(n)}) = \sum_{i=1}^{n} f^i(a_r^i) \qquad \forall \; a_r^i \in \mathcal{A}^i, \; i \in \{1, \; .., n\} \tag{3.5-2}$$

where the f^i's assigned on each dimension are ordinal scales that obey §2.3-2:

$$a_f^i \succ a_g^i \Rightarrow f^i(a_f^i) > f^i(a_g^i)$$

$$a_f^i \; ? \; a_g^i \Rightarrow f^i(a_f^i) = f^i(a_g^i)$$

exists when $\mathcal{A}^{(n)}$ is a K-structure

PROOF: The proof will be inductive, we assume that the theorem holds for an $n-1$ dimensional K-structure, $\mathcal{A}^{(n-1)}$ and show that this can ensure the existence of an additive conjoint scale on $\mathcal{A}^{(n)}$ As §2.3-1 has already proved the existence of an ordinal scale on $\mathcal{A}^1$, this would prove the theorem

From 𝒦𝒫2,

$$a_p^{(n)} \circ a_q^{(n)} \Leftrightarrow [a_i^n \circ a_j^n](1 - \delta_{ij}) \; \vee \; [a_k^{(n-1)} \circ a_l^{(n-1)}](\delta_{ij})$$

We observe that two, and only two, conditions permit the occurrence of strict preference between $a_p^{(n)}, a_q^{(n)} \in \mathcal{A}^{(n)}$

- $a_p^{(n)} \succ a_q^{(n)} \Leftarrow a_k^{(n-1)} \succ a_l^{(n-1)}, \quad i = j$

  By §2.3-2, $a_i^n \equiv a_j^n \Rightarrow f^n(a_i^n) = f^n(a_j^n)$ As we assume that the theorem holds for $\mathcal{A}^{(n-1)}$, we have that

  $a_k^{(n-1)} \succ a_l^{(n-1)} \Rightarrow f^{(n-1)}(a_k^{(n-1)}) > f^{(n-1)}(a_l^{(n-1)})$

  Thus, $f^{(n)}(a_p^{(n)}) > f^{(n)}(a_q^{(n)})$ because

$$f^n(a_i^{\,n}) + f^{(n-1)}(a_k^{\,(n-1)}) \;>\; f^n(a_j^{\,n}) + f^{(n-1)}(a_l^{\,(n-1)})$$

- $a_p^{\,(n)} \succ a_q^{\,(n)} \;\Leftarrow\; a_i^{\,n} \succ a_j^{\,n}\,, \quad i \neq j,$

  This case is to be viewed with care The above expression means that $a_p^{\,(n)} \succ a_q^{\,(n)}$ so long as $a_i^{\,n} \succ a_j^{\,n}$ *even if* it happens that $a_k^{\,(n-1)} \prec a_l^{\,(n-1)}$ Under these conditions, we would have $f^{(n-1)}(a_k^{\,(n-1)}) < f^{(n-1)}(a_l^{\,(n-1)})$ and this might possibly result in a violation of the scale when

  $$f^{(n-1)}(a_l^{\,(n-1)}) - f^{(n-1)}(a_k^{\,(n-1)}) \;>\; f^n(a_i^{\,n}) - f^n(a_j^{\,n})$$

  To avert this eventuality, we introduce the following stipulation We permit only those ordinal assignments on $\mathcal{A}^n$ that ensure that

  $$\left|f^{(n-1)}(a_l^{\,(n-1)}) - f^{(n-1)}(a_k^{\,(n-1)})\right| \;<\; \left|f^n(a_i^{\,n}) - f^n(a_j^{\,n})\right| \qquad \forall \;\; a_i^{\,n}, a_j^{\,n} \in \mathcal{A}^n \qquad \forall \;\; a_k^{\,(n-1)}, a_l^{\,(n-1)} \in \mathcal{A}^{(n-1)}$$

  This requirement is *always* met if we enforce the rule

  $$\left|f^n({}_g a_i^{\,n}) - f^n({}_{g+1} a_j^{\,n})\right| \;>\; \left|f^{(n-1)}({}_1 a_l^{\,(n-1)}) - f^{(n-1)}({}_{\lambda(n-1)} a_k^{\,(n-1)})\right| \qquad \forall \;\; g \in \{1, \quad , \lambda^n - 1\}$$

  Assuming, henceforth, that only the class of assignments that meet the above stipulation are considered, we find that 83 5-2 is always satisfied for $\mathcal{A}^{(n)}$

The rest of the proof follows by induction □

Finally, with enough data available on K-structures in general, it becomes possible - and interesting - to look at the structural properties, this time not of the K-structures themselves, but of the closed class of lattices that qualify as K-structures along with the operation of K-multiplication

THE STRUCTURE OF $\langle \mathfrak{K}, \oplus, \mathfrak{E} \rangle$

The class of K-structures, along with the K-product operation, is closed by definition, is non-commutative ($\mathfrak{T}$3 3-1), associative ($\mathfrak{T}$3 3-2), and has an identity element $\mathfrak{E}$ ($\mathfrak{T}$3 3-6) Thus, it is a ***non-commutative semi-group***

We may finally take an overall look at the contents of this Section Beginning with an introduction to conventional additive conjoint measurement, we drew upon, and extended the results of the previous Section to obtain the representation theorem for additive conjoint measurement Perhaps the most attractive feature of the results is that they reduce to the existing theory when the indeterminancy in the relational system is eliminated This allows us to view the present results as a generalization, possible under certain conditions, of the existing theory

The essence of the results obtained may be interpreted in more than one useful way The obvious interpretation is that the scheme presented premits additive conjoint measurement over incomplete orders that are of the type that we have arbitrarily named the class of ***K-structures***. The other interpretation is that *whenever our physical problem is a K-structure, the theory provides a purely <u>analytical</u> technique for dimensioning the system* To the best of our knowledge, no such procedure has been mentioned in the literature All known methods involve a study of the actual physical problem followed by a procedure to identify a set of gross attributes, these are further dimensioned to yield more attributes until the problem can be decomposed no further In the process, care is to be taken to maintain the property of independence between dimensions as this property is a

prerequisite for additive conjoint measurement For K-structures, on the other hand, the process of inducing a product structure is totally unencumbered by the actual physical situation (the actual physical problem is quite irrelevant to the measurement problem) and moreover, independence is inherently preserved

The discovery at the end of our investigations that the class of K-structures constitute a semi-group directs us to ask whether at all the possibility exists of *analysing* K-structures The next Section addresses that problem and provides a crucial tool - an algorithm for the progressive *decomposition* of a K-structure

SECTION 4

The Decomposition Algorithm

4.1 REVIEW: THE NEED FOR AN ALGORITHM

The previous Section ended with the conclusion that the class of K-structures is a non-commutative semi-group The direct implication of this is that ~~no~~ members of the class may ha~~s~~ve ~~an~~ no inverse element, the product with which can yield the identity element The disadvantage that results from this is that there is no simple way by which we can determine the operands of a given multiple K-product Even if only to get some interesting insights, it is worthwhile to launch on an otherwise entirely fruitless discussion of the (completely hypothetical) situation that would prevail had inverses existed Then we could, knowing $\mathcal{A}^n$ (and therefore, also its inverse) easily regenerate the K-multiplicand $\mathcal{A}^{(n-1)}$ by computing the product $\overline{\mathcal{A}^n} \otimes \mathcal{A}^{(n)}$ where

$$\overline{\mathcal{A}^n} \otimes \mathcal{A}^n = 8 = \mathcal{A}^n \otimes \overline{\mathcal{A}^n}$$

In the above notation, $\overline{\mathcal{A}^n}$ is clearly what is generally known as the inverse of $\mathcal{A}^n$ Now, $\mathcal{A}^n$ is a linearizable structure - a K-structure of dimension unity As the K-product is a non-commutative operation, the inverses of structures of

higher dimensions are related to the inverses of their component dimensions in a manner identical to that encountered in the context of matrix multiplication because, the matrix product, too, is non-commutative. Thus, if

$$\mathcal{A}^{(n)} = \mathcal{A}^{n} \otimes \mathcal{A}^{n-1} \otimes \quad \otimes \mathcal{A}^{1},$$

its inverse, had it existed, would be

$$\overline{\mathcal{A}^{(n)}} = \overline{\mathcal{A}^{1}} \otimes \quad \otimes \overline{\mathcal{A}^{n-1}} \otimes \overline{\mathcal{A}^{n}}$$

It may also be seen that the left and right inverses would be the same

$$\overline{\mathcal{A}^{(n)}} \otimes \mathcal{A}^{(n)} = \mathcal{A}^{(n)} \otimes \overline{\mathcal{A}^{(n)}} = \mathcal{E}$$

With that brief diversion we return to reality which asserts that K-structures do not have inverses. The assertion can easily be made respectable with a somewhat more formal proof: we know that the cardinality of a K-structure (of *any* set, for that matter) has to be a positive integer; we also know that the cardinality of a K-product is the product of the operands' cardinalities. As the cardinality of the identity element is unity, (§3.3-6) we are led to conclude that if $\overline{\mathcal{A}^{(n)}}$ existed, then its cardinality

$$\left\|\overline{\mathcal{A}^{(n)}}\right\| = \left[\left\|\mathcal{A}^{(n)}\right\|\right]^{-1}$$

which can never be an integer, except when $\left\|\mathcal{A}^{(n)}\right\|$ is itself unity, which can happen only when $\mathcal{A}^{(n)} = \mathcal{E}$ - which would amount to the tautology

$$\mathcal{E} = \mathcal{E} \otimes \overline{\mathcal{E}}$$

The proof given above points to the need for at least an algorithm for purposes of *analysis* of K-structures. An algorithm that can progressively decompose a K-structure has been found. The fact that a decomposition algorithm exists has another implication that is of consequence. This is that we are able, on the strength of the algorithm, to frame an alternative definition

for a K-structure that is better suited for *identification* of a K-structure than that given at the beginning of Subsection 3 2 which, while being constructive, was incapable of helping us to determine whether a given structure was a K-structure

K-STRUCTURE: ALTERNATIVE DEFINITION

A given structure is a K - structure, if, upon application of the K- structure decomposition algorithm, it

- ***does not cause the algorithm to fail, and***
- ***terminates the algorithm after a finite number of iterations***

At the end of each iteration, the algorithm frees the "outermost" dimension of the structure *and* exposes the quotient structure, this means that we are given the one-dimensional K-multiplier $\mathcal{A}^{n}$ and the K-multiplicand (the "quotient") $\mathcal{A}^{(n-1)}$ after one iteration of the algorithm on $\mathcal{A}^{(n)}$ The next iteration will be on $\mathcal{A}^{(n-1)}$, and it will yield $\mathcal{A}^{n-1}$ and $\mathcal{A}^{(n-2)}$ The final iteration will therefore yield $\mathcal{A}^{2}$ and $\mathcal{A}^{1}$ In effect, the algorithm produces the constituent dimensions of the structure in the reverse order of their composition The next Subsection presents the algorithm itself

4.2 *THE DECOMPOSITION ALGORITHM*

We denote the *ordered* set of levels of the K-structure $\mathcal{A}^{(n)}$ (as are obtained by the use of the level generating algorithm $\mathcal{A}$3 4-1 given at the start of Subsection 3 4) by $L = \{\ell_1^{(n)}, \quad , \ell_{\lambda(n)}^{(n)}\}$ The cardinality of $\ell_i^{(n)}$ is $|\ell_i^{(n)}|$ New notations are explained as they are introduced.

The principle on which the algorithm operates is first explained Assuming that the K-multiplier $\mathcal{A}^n$ is not the trivial case of a linear structure (which means that we have $? \neq \emptyset$ on $\mathcal{A}^n \times \mathcal{A}^n$), we are then assured that there exists at least one level $\ell_*^n \subset \mathcal{A}^n$ s t $|\ell_*^n| > 1$ If, in any problem, it happens that $\mathcal{A}^n$ is linear, then the algorithm considers $\mathcal{A}^n \otimes \mathcal{A}^{n-1}$ to be the K-multiplier rather than just $\mathcal{A}^n$ alone In general, the rule is that the algorithm views as the K-multiplier as many operands on the left of the product as are necessary to constitute a structure with at least one $?$ relation in it If there exist more than one level of width greater than unity, the algorithm chooses the level with the largest cardinality - in any case, we wish to assure ourselves that we capture a level containing more than one element so that $?$ is not empty in that level For any ${}_*a_i^n,\ {}_*a_j^n \in \ell_*^n \subset \mathcal{A}^n$, the substructures ${}^i\mathcal{A}^{(n)},\ {}^j\mathcal{A}^{(n)}$ generated by the elements will possess the mutual property

$$ {}_ia^{(n)} \mathrel{?} {}_ja^{(n)} \qquad\qquad \forall\ {}_ia^{(n)} \in {}^i\mathcal{A}^{(n)},\ \ \forall\ {}_ja^{(n)} \in {}^j\mathcal{A}^{(n)} $$

The algorithm locates the entire *layer* (a layer of substructures is the set of all substructures generated by all the elements of any one level ℓ_h^n of $\mathcal{A}^n$) of substructures generated by all the ${}_*a_i^n \in \ell_*^n$ and names it ℓ_*^* The set of maxima $M = \{m_i\}$ of ℓ_*^* is identified (it is just the union of the sets of maxima of the individual ${}^i\mathcal{A}^{(n)}$ Next, we use the property of ℓ_*^* that the restriction of the relation $(\succ \cup \equiv \cup \prec)$ in it forms an equivalence relation - it may be checked that this is so by observing that reflexivity, symmetry and transitivity hold The resulting equivalence classes are isomorphisms of one another and are indeed the substructures themselves and the cardinality of the partition, henceforth called the *width* of that layer, is the cardinality of ℓ_*^n Obviously, therefore, we effect such a partitioning on ℓ_*^* Any one member of

the partition, along with the restriction of $\succ$ and ? defined on it is a substructure ${}'\mathcal{A}^{(n)}$ and we know that this is isomorphic to $\mathcal{A}^{(n-1)}$ (by the incidental discussions in §3 4-3) This solves the problem of finding the K-multiplicand We are now only left with the problem of finding the K-multiplier $\mathcal{A}^{n}$ Two neighbouring levels $\ell_{i}^{(n)}$ and $\ell_{i+1}^{(n)}$ of $\mathcal{A}^{(n)}$ always constitute a ?-transitive structure whenever they are the last level of one substructure layer and the first level of the next substructure layer, respectively, in other words, whenever they constitute the interface between two neighbouring substructure layers (this can be shown to follow from 𝔅2 3-5 and 𝔎𝔓2) Such ?-transitive level pairs are identified right at the start of the algorithm Since other ?-transitive level pairs may also exist (*within* layers of unit width), we identify those particular level pairs that actually signify a layer transition by assessing the cardinality of the entire structure sandwiched between any two neighbouring ?-transitive pairs, if the abovementioned included structure really constitutes a whole layer, its cardinality will be a multiple of $\eta^{(n-1)}$, the cardinality of $\mathcal{A}^{(n-1)}$ Once the different successive layers are thus identified, simply dividing the cardinality of each by $\eta^{(n-1)}$ yields the number of substructures in that layer which, by the discussions in connection with §3 4-3, is just the cardinality of the level in $\mathcal{A}^{n}$ that generated it Thus, eventually, we will be able to reconstruct the array representation of $\mathcal{A}^{n}$ in the form $[|\ell_1{}^{n}|, \quad , |\ell_{\lambda n}{}^{n}|]$, which as has already been argued in connection with its introduction, is sufficient to specify a linearizable structure to within an automorphism Thus, $\mathcal{A}^{n}$ has also been found In the algorithm that follows, use will be made of the already given linearization ($\mathcal{A}$2 4-2), ?-transitivity test ($\mathcal{A}$2 4-1), and the level generation ($\mathcal{A}$3 4-1) algorithms on various occasions

ALGORITHM 4.2-1 DECOMPOSITION

Begin { Decomposition }

Set $n = 1$,

1 $n = n - 1$

Partition $\mathcal{A}^{(n)}$ with $\mathcal{A}3$ 4-1 to get the ordered partition

$L = \{\ell_1^{(n)}, \quad , \ell_{\lambda(n)}^{(n)}\}$

2 Set $i = 1$, $k=1$,

For $i \in \{1, \quad , \lambda^{(n)}-1\}$, perform $\mathcal{A}2$ 4-1 on $\ell_i^{(n)} \cup \ell_{i+1}^{(n)}$

If ?-transitivity is true then

$\ell_i^{(n)} \in L' \triangleq \{\ell'_1, \quad , \ell'_k, \quad , \ell'_{n'}\}$

index $(k) = i$

$k = k + 1$

3 Form $L^* \triangleq \{\ell^*_1, \quad , \ell^*_p, \quad , \ell^*_{n*}\}$ as follows

$$\ell^*_1 \triangleq \bigcup_{i=1}^{ind(1)} \ell_i^{(n)},$$

$$\ell^*_p \triangleq \bigcup_{i=ind(p-1)+1}^{ind(p)} \ell_i^{(n)}, \qquad \forall \; p \in \{2, \; . \; , n^*-1\},$$

$$\ell^*_{n*} \triangleq \bigcup_{i=ind(n')+1}^{\lambda^{(n)}} \ell_i^{(n)}$$

4 Find $\ell^*_s \in L^*$ s t $|\ell^*_s| \geq |\ell^*_p|$, $\qquad \forall \; \ell^*_p \in L^*$

5 Find $M = \{m_1, \quad , m_e\}$ s t $\{x \mid (x \in \ell^*_s) \wedge (x \succ m_j)\} = \emptyset$

$\forall \; j \in \{1, \quad , e\}$

6 Find $\xi(m_j) \triangleq \{x \mid x \succeq m_j\}$ $\qquad \forall \; j \in \{1, \quad , e\}$

7 $\Phi = \emptyset$

For $j \in \{1, \quad , e\}$

if $(\xi(m_1) \cap \xi(m_j)) \neq \emptyset$, then

$\Phi = (\xi(m_1) \cup \xi(m_j) \cup \Phi)$

8 Find $\frac{|\ell^*_s|}{|\Phi|} \triangleq a_q^n$

If $m = n^*$, go to 14

9 Repeat (10) while $(m+r) \leq n^*$

Set $r = 1$, $q = 1$,

$\varsigma \triangleq \ell^*_{s+r}$

10 If $\frac{|\varsigma|}{|\Phi|} < 1$, then go to 11

else go to 12

11 $r = r+1$

$\varsigma = \varsigma \cup \ell^*_{s+r}$

go to 10

12 $a_q^n = \frac{|\varsigma|}{|\Phi|}$

$q = q+1$

$r = r+1$

$\varsigma \triangleq \ell^*_{s+r}$

go to 10

14 Repeat (15) while $(m+r) > 0$

If $m = 1$, go to 18

Set $s = -1$, $q = -1$

$\varsigma \triangleq \ell^*_{s+s}$

15 If $\frac{|\varsigma|}{|\Phi|} < 1$, then go to 16

else go to 17

16 $\imath = \imath - 1$

$\varsigma = \varsigma \cup \ell^*_{s+r}$

go to 15

17 $a_q^{\,n} = \frac{|\varsigma|}{|\Phi|}$

$q = q-1$

$r = r-1$

$\varsigma \triangleq \ell^{*}_{n+r}$

go to 15

18 $\mathcal{A}^{n} = \{a_p^{\,n}\}$

$\mathcal{A}^{(n-1)} = \Phi$

19 Perform $\mathcal{A}$2 4-1 on $\mathcal{A}^{(n-1)}$

If ?-transitivity is false, then

$\mathcal{A}^{(n-1)} \triangleq \mathcal{A}^{(n)}$

go to 1

End { Decomposition }

This Section is concluded here with a few remarks concerning the decomposition algorithm A close look will show that the algorithm assumes that the structure it operates on possesses some basic properties that characterize a K-structure If the structure does not have these properties, the algorithm will fail The failure of the algorithm may itself be taken as an indication that the given structure is not a K-structure. In this sense, the above algorithm may itself be used indirectly as a testing algorithm $\mathfrak{F}$4 2-1*a* and $\mathfrak{F}$4 2-1*b* diagramatically represent some of the operations carried out on the K-structure at successive stages of the algorithm

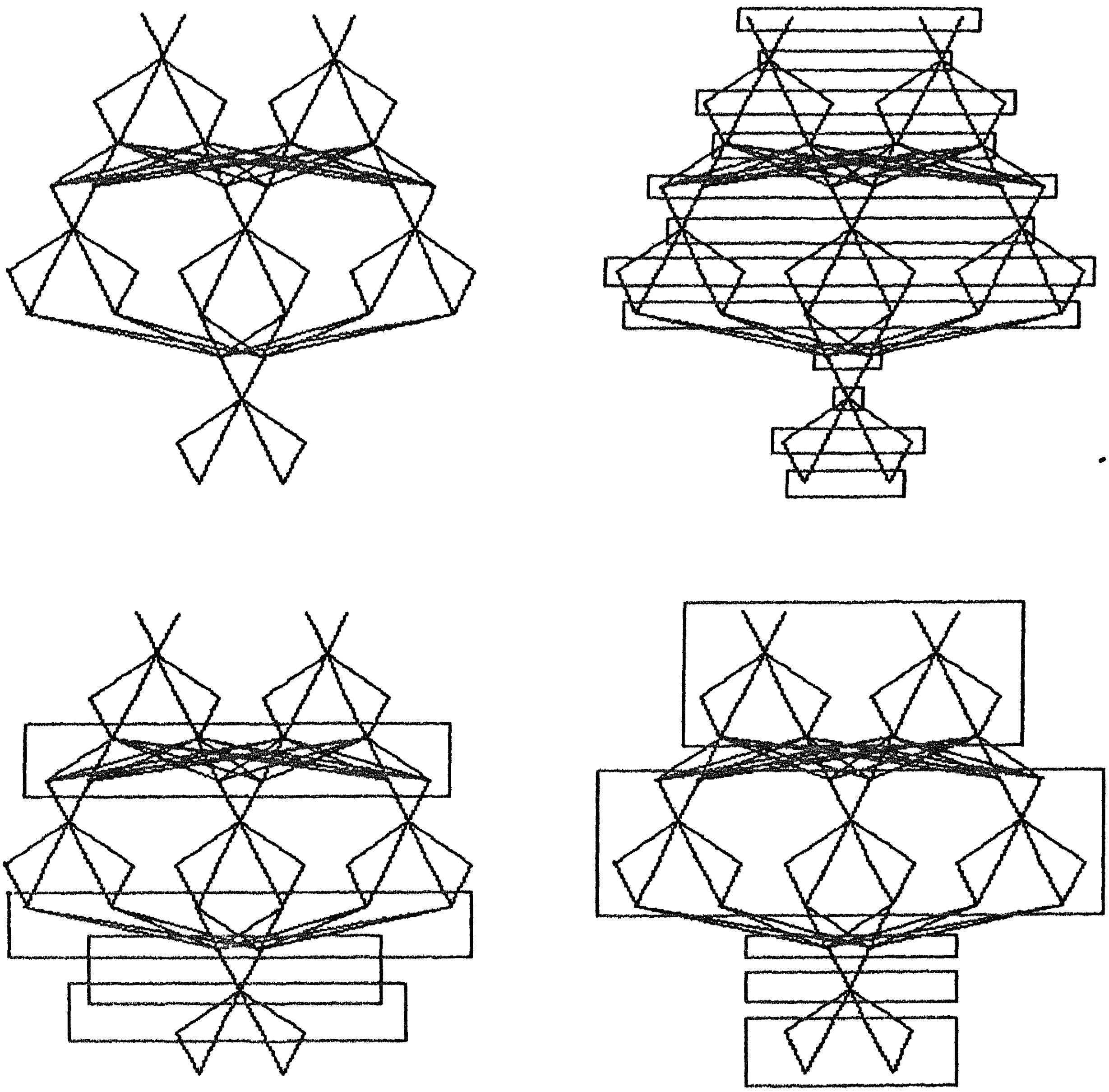

Fig.4.2-1a: Decomposition Algorithm

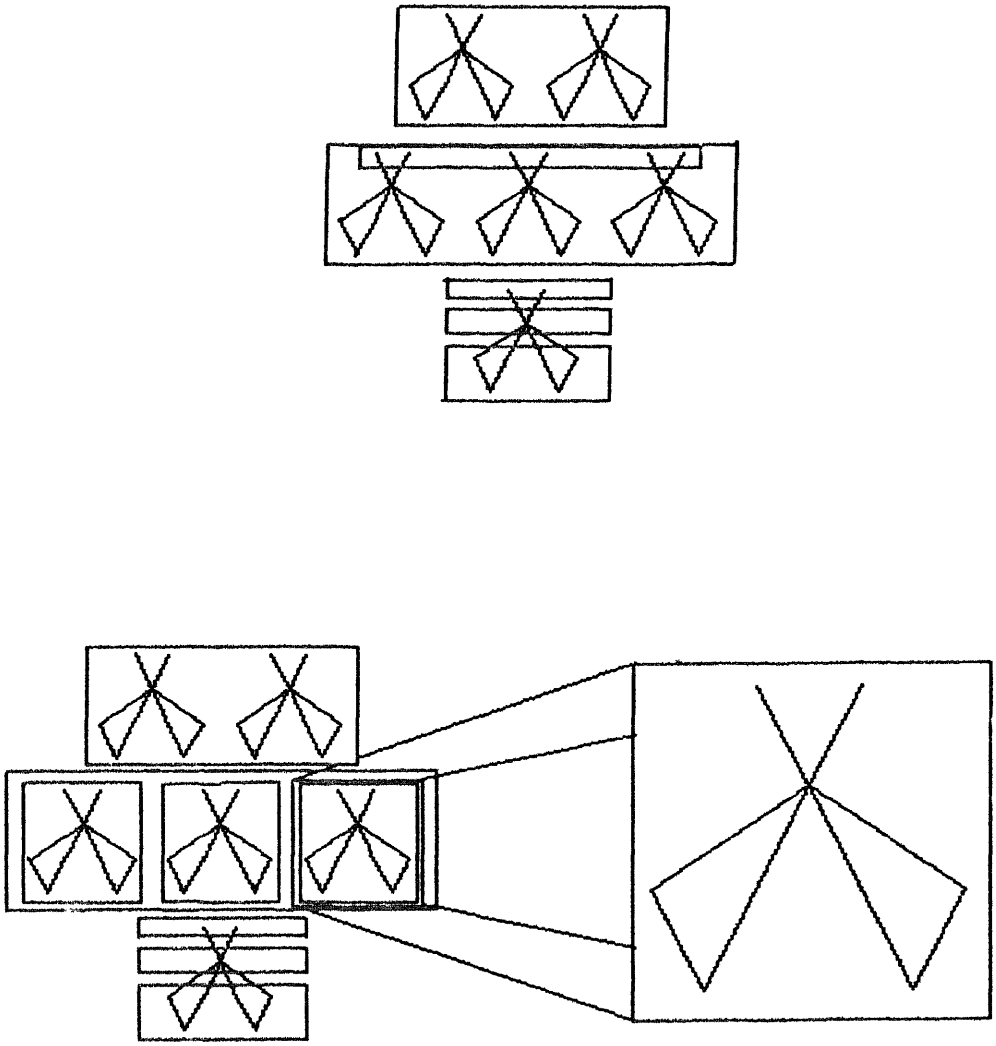

Fig.4.2–1b: Decomposition Algorithm (cont'd.)

SECTION 5

Conclusion

The primary objective of this thesis was to attempt ordinal, or sub-ordinal measurement over incompletely ordered relations We are now in a position to view critically the entire body of results that have been obtained and compare what has actually resulted with what was intended The points of interest may be summarized as follows

- We find that our solution applies only to a specific subclass of the class of incomplete structures - those of the ?-transitive type The results obtained even go to show that the present approach is definitely unsuitable for incomplete orders that are not ?-transitive (¶2 3-1)
- The other point of interest is that, under the assumption of a variant of the Laplacean indifference principle, our measurement scale is as strong as the conventional ordinal scale meant for weak orders The fact that partial orders in general are inherently invested with a less rich structure than are weak orders does not seem to have manifested itself in the structural sophistication of the measurement scale we created

The first point raised above directs us to ask what the purely *physical* implications are of the ?-transitivity assumption Does this assumption, in some as yet unclear sense, correspond to a physically trivial case of the general indeterminate situation? It is important to clarify here that we are not hoping that, at some time in the future, a new theoretical formulation will enable us to implement ordinal measurement on the general incomplete order That is inherently impossible for an incomplete order (indeed, *because* it is incomplete) and no amount of future theory can change this fact

What we *can* hope to achieve is a totally new measurement scale that would naturally be less structured than the ordinal scale, but will be able to encompass the entire class of incomplete orders That this seems necessary is hinted by the second point raised above In spite of starting with a *less structured* physical problem than that which the ordinal scale is meant to handle, we were able to get away with a scale that is more sophisticated than the quantity Such a scale assignment ought to have spelled havoc by bringing in the classical problem of *overmeasurement* - a case of the scale saying more than the physical problem itself does This did not happen because of the assumption of our version of the Principle of Insufficient Reason As was stated when this assumption of the decisionmaking criterion was introduced, the stand taken can be supported only at the level of plausible argument - not at the logical level Indeed, it is because there may exist several other decision criteria (possibly also accompanied by arguments just as convincing as those we have ourselves offered) that our formulation will only occupy an equal status with other future attempts of a similar kind that do not try to conceive of a totally new scale type meant especially for handling the general

incompletely ordered system Such a general approach must base its theory entirely on logical grounds and not fall back on this or that decision criterion (or, what is no better, invent new decision criteria of its own)

At the present time, it is not clear which domains of study contain physical problems that conform directly to the linearizable structure model - or even the more general K-structure model The degrees of success that are achieved by future attempts to apply the present theory alone can tell how useful the theory is, *and in exactly what way* Future application will also point to the real-life consequences of the ?-transitivity assumption and the indifference principle

Considering the theory for its own sake for a moment, we see that it at least contains no internal contradictions and it is, therefore, logically consistent So, as things stand at present, we may claim to have produced a logically consistent theory that is yet to see practical application

References

Campbell, N R 1920, *Physics The Elements*, Cambridge University Press, Cambridge, Massachusetts

Campbell, N R 1928, *An Account of the Principles of Measurement and Calculation*, Longmans, Green, London

Cox, R T 1961, *Algebra of Probable Inference*, Baltimore, Johns Hopkins Press

Keeney, R L and Raiffa, H 1976, *Decisions With Multiple Objectives Preferences and Value Tradeoffs*, John Wiley, New York

Luce, R D and Tukey, J W 1964, 'Simultaneous Conjoint Measurement A New Type of Fundamental Measurement', *J Math Psychol*, **1**, 1-27

Milnor, J 1954, 'Games Against Nature' in R M Thrall, C H Coombs, and R L Davis, (eds), *Decision Processes*, Wiley, New York pp 49-59

Narens, L 1985, *Abstract Measurement Theory*, The MIT Press, Cambridge, Massachusetts

Roberts, F S 1979, *Measurement Thepry*, Addison Wesley Publishing Company

Roberts, F S and Franke, C H 1976, 'On The Theory of Uniqueness in Measurement', *J Math Psychol*, **14**, 211-218

Scott, D 1964, 'Measurement Models and Linear Inequalities', *J Math Psychol*, **1**, 233-247

Stevens, S S 1957, 'On the Psychophysical Law', *Psychol Rev*, **64**, 153-181

Suppes, P and Scott, D 1958, 'Foundational Aspects of Theories of Measurement', *J Symb Logic*, **23**, 113-128

Suppes, P 1961, 'The Philosophical Relevance of Decision Theory' *Journal of Philosophy*, **58**, 605-614

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